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Théorèmes de Liouville et singularités dans les équations aux dérivées partielles

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Introduction générale

1 Singularités dans les équations d'évolution

Un nombre important de problèmes en sciences appliquées peut être modélisé par des systèmes d'équations différentielles ordinaires (EDO) ou des systèmes d'équations aux dérivées partielles (EDP). Ces équations sont souvent non linéaires. Elles ont des propriétés complètement différentes de la théorie linéaire. Elles sont de ce fait plus difficiles et plus riches à étudier.

Une des propriétés les plus importantes qui a distingué les problèmes d'évolution non linéaires est l'éventuelle formation de singularités en temps fini, pour des solutions provenant de données régulières. On note que même si les singularités existent dans les problèmes linéaires, elle sont dues aux singularités contenues dans les coefficients ou les données du problème. En revanche, dans les systèmes non linéaires, les singularités peuvent provenir du mécanisme non linéaire du problème.

Donnons dès maintenant un exemple de singularités pour les EDO, avant d'aborder le cas des EDP.

1.1 Explosion dans les équations différentielles ordinaires

Considérons l'EDO suivante

$$u' = f(u), \quad u(0) = a \quad \text{et } f \in C^1(\mathbb{R}). \tag{1}$$

D'après le théorème de Cauchy-Lipschitz, on a l'existence et l'unicité d'une solution maximale. Si $T > 0$ est le temps maximum d'existence de la solution, deux cas se présentent :

- $T = +\infty$: on parle d'existence globale. C'est le cas lorsque f est linéaire, mais également dans certains cas où f est non linéaire (ex : $u' = u - u^2$, $u(0) = a$ où la solution est donnée par $u(t) = a/\left(a - (a-1)e^{-t}\right)$, bien définie pour tout $t \geq 0$).
- $T < +\infty$: dans ce cas $\lim_{t \to T} |u(t)| = +\infty$ et on dit que u *explose* en temps fini T. On dit aussi que u admet une *singularité* au temps T. C'est le cas de

$$u' = u^2, \quad t > 0, \quad u(0) = a.$$

En effet, pour une donnée $a > 0$, il est immédiat qu'une solution unique existe pour $0 < t < T = 1/a$. Sachant que la solution est donnée par la formule $u(t) = 1/(T - t)$,

on remarque qu'elle est régulière pour $t < T$ et $u(t) \to \infty$ quand $t \to T^-$ (la limite à gauche).
Le concept d'*explosion* peut être géneralisé à un phénomène pour lequel les solutions cessent d'exister globalement en temps à cause de la croissance de la variable décrivant le processus d'évolution. Plus généralement, une condition suffisante d'explosion pour (1) est donnée par

$$f \geq 0 \text{ et } \int_1^\infty ds/f(s) < \infty,$$

comme c'est le cas pour $u' = u^p$, avec $p > 1$.

1.2 Explosion dans les équations aux dérivées partielles

Au cours de cette thèse, on s'est interessé à la formation de singularités en temps fini pour deux familles d'EDP non linéaires :
- les équations de type chaleur,
- les équations de type ondes.

Si au niveau de la partie linéaire, ces équations sont très différentes (vitesse de propagation finie pour les ondes, effet régularisant pour la chaleur,...), le terme non linéaire les rapproche. En effet, chacune présente le phénomène d'apparition de singularités en temps fini et leur traitement se base sur des approches similaires (Fonctionnelle de Lyapunov, Théorèmes de type Liouville,...) comme nous allons le voir dans la suite.

Présentons maintenant les deux familles d'équations.

Équations semi-linéaires de type chaleur :
On considère le système de reaction-diffusion suivant

$$\begin{cases} \partial_t u &= \Delta u + F(u) \quad \text{dans } \mathbb{R}^N \times [0,T), \\ u(.,0) &= u_0 \quad \text{dans } \mathbb{R}^N, \end{cases} \tag{2}$$

où $u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}^M$, $u_0 : \mathbb{R}^N \to \mathbb{R}^M$, $F : \mathbb{R}^N \to \mathbb{R}^M$ est de classe C^1 et $N, M \in \mathbb{N}$.
Ce système constitue un modèle simplifié pour beaucoup de phénomènes physiques de réaction-diffusion. Il apparait notamment en combustion (voir Kapila [Kap80], Kassoy et Poland [KP80], [KP81] (explosion thermique), Bebernes et Eberly [BE89], Galaktionov, Kurdyumov et Samarskii [GKS84], Galaktionov et Vazquez [GV93], [GV02]). On le retrouve aussi dans beaucoup de situations physiques, de la mecanique des fluides à l'optique, sous la forme de l'équation de Ginzburg-Landau complexe (voir Levermore et Oliver [LO96]). Le système (2) a aussi un grand inérêt en biologie, en particulier pour les modèles de dynamique des populations (voir Gaucel et Langlais [GL02] et [GL07], Hamel et Roques [RH07], Berestycki et Rossi [BR08], Aronson et Weinberger [AW78]), ainsi que pour la transmission de l'influx nerveux (voir Nagasawa [Nag68], McKean [McK75]).
Le problème de Cauchy (local en temps) pour (2) peut être résolu dans une grande classe d'espaces fonctionels. On peut alors définir $T > 0$ comme étant le temps maximum d'existence de la solution de (2). Deux cas se présentent alors :

- $T = +\infty$: on parle d'existence globale.
- $T < +\infty$. On dit que u solution de (2) explose en temps fini. Dans ce cas,

$$\lim_{t \to T} \|u(t)\|_{L^\infty} = +\infty.$$

Dans cette thèse, on s'intéresse en particulier à une équation où $\frac{1}{u}$ explose en temps fini. On parle alors d'*extinction* en temps fini pour u (voir sous section 2.3).

Équations semi-linéaires de type ondes :
On considère également l'équation des ondes semi-linéaire suivante :

$$\begin{cases} \partial^2_{tt} u &= \partial^2_{xx} u + |u|^{p-1} u, \\ u(0) &= u_0 \text{ et } \partial_t u(0) = u_1, \end{cases} \tag{3}$$

où $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in H^1_{loc,u}$ et $u_1 \in L^2_{loc,u}$ avec

$$\|v\|^2_{L^2_{loc,u}} = \sup_{a \in \mathbb{R}} \int_{|x-a|<1} |v(x)|^2 \, dx \text{ et } \|v\|^2_{H^1_{loc,u}} = \|v\|^2_{L^2_{loc,u}} + \|\nabla v\|^2_{L^2_{loc,u}}.$$

Le problème de Cauchy pour l'équation (3) dans l'espace $H^1_{loc,u} \times L^2_{loc,u}$ découle de la vitesse de propagation finie et de la solution du problème de Cauchy dans $H^1 \times L^2$ (voir Ginibre, Soffer et Velo [GSV92], Shatah et Struwe [SS93a], Lindblad et Sogge [LS95]). Deux cas se présentent alors (voir Alinhac [Ali02] et [Ali95]) :

- soit $u(x,t)$ est définie pour tout $(x,t) \in \mathbb{R} \times [0, +\infty)$. La solution est dite globale.
- soit il existe un graphe $(x \mapsto T(x))$ 1−Lipschitzien tel que u est définie sous le graphe et ne peut être étendue au delà. Le graphe de T est appellé l'ensemble d'explosion. Contrairement au cas de l'équation de la chaleur où le temps d'explosion est unique, il existe pour l'équation des ondes un temps local d'explosion pour chaque $a \in \mathbb{R}$. Cette différence vient de la vitesse de propagation qui est finie ($= 1$) pour les ondes et infinie pour la chaleur.

1.3 Directions de l'étude des singularités

Parmi les problèmes posés dans la littérature, on peut distinguer deux grandes problématiques :

- **Construction :** Il s'agit de construire des exemples de solutions qui explosent en temps fini, éventuellement avec prescription d'un certain comportement asymptotique. Autrement dit, il s'agit de chercher des conditions suffisantes sur la donnée initiale ou sur le terme non linéaire pour avoir explosion.
- **Description :** Considérant une solution explosive quelconque, on se demande quel sera son comportement asymtotique au voisinage du (ou des) temps d'explosion.

Notre thèse est constituée de 4 papiers, dont 3 sont acceptés pour publication. On distinguera deux parties :

- La première partie concerne l'étude de la formation de singularités (explosion et extinction) en temps fini pour l'équation de la chaleur semi-linéaire.
- Dans la deuxième partie, on établit la régularité $C^{1,\alpha}$ de la courbe d'explosion aux points non caractéristiques pour une équation des ondes semi-linéaire.

2 Équation de la chaleur semi-linéaire

On vise dans cette partie de la thèse l'étude de la formation de singularités en temps fini dans des systèmes de réaction-diffusion de type chaleur. Pour fixer les idées, prenons l'exemple prototype de la littérature, donné par :

$$\begin{array}{rcl} \partial_t u &=& \Delta u + |u|^{p-1}u. \\ u(.,0) &=& u_0, \end{array} \tag{4}$$

où $u(t) : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ et $N \in \mathbb{N}$. On suppose aussi que p est sous-critique dans le sens suivant :

$$p > 1 \text{ et } p < (N+2)/(N-2). \tag{5}$$

Notons que le cas où p est surcritique ($N \geq 3$ et $p > (N+2)/(N-2)$) a fait l'objet de plusieurs publications : Matano et Merle [MM08] et [MM04], Mizoguchi et Senba [MS07], Matos [Mat01], [Mat99] et Quittner et Souplet [QS07].
Le problème (4) avec p sous-critique a attiré beaucoup d'attention. En effet, il a plusieurs points communs avec beaucoup de problèmes d'explosion qui proviennent de la physique (comme le rôle de chagement d'échelle ou des variables auto-smilaires). On cite parmi ces problèmes : le mouvement par courbure moyenne (Soner et Souganidis [SS93b]), la dynamique des vortex dans les supraconducteurs de type II (Chapman, Hunton et Ockendon [CHO98], Merle et Zaag [MZ97]), la diffusion de surface (Bernoff, Bertozzi et Witelski [BBW98] et la chémotaxie (Brenner et al [BCK+99], Herrero et Velázquez [HV97], Perthame [Per04]). L'intérêt de l'équation (4) est qu'elle conserve des propriétés fondamentales des modèles physique, tout en restant simple pour l'étude mathématique.
Sur cette équation, on va revoir l'historique (non exhaustif vue la taille de la littérature) des questions de construction et de description posées en Section 1.
- **Construction :** Beaucoup d'auteurs ont cherché des conditions suffisantes pour l'explosion pour l'équation (4). Nous avons choisi de n'en présenter que 3 dans la suite.

- Dans le cas d'un domaine borné Ω, Ball [Bal77], (voir aussi Levine [Lev73]) a obtenu grâce à l'énergie associée à (4)

  $$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx,$$

  et à des méthodes d'équations différentielles ordinaires, une condition suffisante pour l'explosion d'une solution de (4) :

  Si $u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$ et $E(u_0) \leq 0$, alors $u(t)$ explose en temps fini.

  Ce résultat est également valable si $\Omega = \mathbb{R}^N$, grâce à un critère d'explosion de Merle et Zaag [MZ00] (voir Matano et Merle [MM08]).
- Dans le papier de Fujita [Fuj66] (voir aussi Senba et Suziki [SS04], pour des cas critiques voir Hayakawa [Hay73], Kobayashi, Sirao et Tanaka [KST77]), l'auteur a démontré que :

  Si $p \leq 1 + \frac{2}{N}$, $u_0 \in C^2 \cap W^{2,\infty}(\mathbb{R}^N)$ et $u_0 \neq 0$, alors l'équation (4) a une unique solution $u(x,t)$ qui explose en temps fini T.

- Kaplan [Kap63] a aussi donné une condition suffisante pour l'explosion en temps fini pour des solutions $u(x,t)$ de l'équation (4) avec condition initiale $u_0 \geq 0$. On donne ici une version de son résultat adaptée à notre équation (voir chapitre 1, Proposition 1.2.1, page 24).

  Il existe $M > 0$ tels que si $u_0(x)$ est dans $L^\infty(\mathbb{R}^N)$, positive et vérifie

$$\forall |x| \leq M, \quad u_0(x) \geq \frac{\kappa}{2}, \text{ où } \kappa = (p-1)^{-\frac{1}{p-1}}$$

  alors la solution $u(x,t)$ de l'équation (4) avec condition initiale V_0 explose en temps fini $T > 0$.

- Description : Une abondante littérature est dédiée à cette question. On cite pami les auteurs : Weissler [Wei84], Giga et Kohn [GK85], [GK87], [GK89], Bricmont et Kupiainen [BK94], Herrero et Velázquez [HV93], Matos et Souplet [MS03]....).
Cependant, la plupart de ces travaux ne permettent pas d'obtenir des estimations uniformes en espace ou par rapport aux données initiales. Ainsi, faute d'estimations uniformes, des questions importantes ne pouvaient être abordées, comme par exemple la stabilité du comportement à l'explosion ou encore la régularité de l'ensemble d'explosion. Une nouvelle approche basée sur la preuve de Théorèmes de Liouville (ou de rigidité) pour des solutions entières, a ouvert la porte à l'obtention de telles estimations uniformes (voir pour l'équation de la chaleur Merle et Zaag [MZ98b], [MZ98a], [MZ00] et Nouaili et Zaag [NZ08], pour l'équation de Korteweg de Vries (Martel et Merle [MM00]), pour l'équation de Schrödinger non linéaire Merle and Raphael [MR04], [MR05] et pour l'équation des ondes (Merle and Zaag [MZ08a], [MZ08b].
Cette approche a motivé la plus grande partie de notre travail. En effet, notre premier travail [Nou08] présente une démonstration simplifiée du Théorème de Liouville de Merle et Zaag [MZ98b], [MZ98a] pour la chaleur dans le cas positif. Dans le deuxième et troisième travail, on tente d'élargir la classe des EDP où l'on obtient un Théorème de Liouville, en nous intéressant à l'EDP complexe suivante :

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1}u,$$

(notez que la non linéarité n'est pas un gradient), ou encore à l'équation de la chaleur suivante avec terme d'absorption :

$$\partial_t u = \partial_x^2 u - \frac{1}{u^\beta}.$$

La suite de cette partie est consacrée à ces 3 travaux.

2.1 Démonstration simplifiée du Théorème de Liouville pour l'équation de la chaleur semi-linéaire

Dans cette sous-section, nous présentons les résultats de notre article publié en 2008 dans le *Journal of Dynamics and Differential Equations* [Nou08].

Dans [MZ98a] et [MZ00], Merle et Zaag considèrent l'équation (4) et montrent le Théorème de Liouville suivant :

Théorème 0.1. *(Merle-Zaag) Supposons (5) et considérons u une solution de (4) défine pour tout* $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. *Supposons en plus que* $|u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$, *pour une constante* $C > 0$. *Alors* $u \equiv 0$ *ou il existe* $T_0 \geq T$ *tel que pour tout* $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \pm\kappa(T_0 - t)^{-\frac{1}{p-1}}$ *avec* $\kappa = (p-1)^{-\frac{1}{p-1}}$.

En introduisant les *variables auto-similaires* suivantes :

$$y = \frac{x}{\sqrt{T-t}},\ s = -\log(T-t),\ w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \tag{6}$$

l'équation (4) est transformée en :

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + |w|^{p-1} w, \tag{7}$$

et on a une autre formulation équivalente du Théorème de Liouville.

Théorème 0.2. *(Merle-Zaag) On suppose (5) et on considère* $w(y,s)$, *une solution bornée de (7), défine pour tout* $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. *Alors* w *est l'une des solutions suivantes :* $w \equiv 0$, *ou* $w = \pm\kappa$, *ou il existe* $s_0 \in \mathbb{R}$, *tels que* $w = \pm\varphi(s - s_0)$ *avec* $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$.

Ce théorème introduit une nouvelle approche dans l'étude de l'équation (4). En effet, il donne des estimations uniformes par rapport à l'espace et/ou à la condition initiale. On cite alors le résultat suivant de [MZ98a] et [MZ00] :

(Comportement de type EDO) Considérons $u(x,t)$ *une solution de l'équation (4) qui explose en temps fini* $T > 0$. *Alors, pour tout* $\epsilon > 0$, *il existe* $C(\epsilon)$ *tel que pour tout* $x \in \mathbb{R}^N$ *et* $t \in [T/2, T)$,

$$\left|\partial_t u(x,t) - |u|^{p-1} u(x,t)\right| \leq \epsilon \left|u(x,t)\right|^p + C.$$

(Comportement de type EDO) Considérons $T \leq T_0$, $\|u_0\|_{C^2(\mathbb{R}^N)} \leq C_0$ *et* $u(x,t)$ *la solution de l'équation (4), avec condition initiale* u_0. *Alors, pour tout* $\epsilon > 0$, *il existe* $C(\epsilon, C_0, T_0)$ *tel que pour tout* $x \in \mathbb{R}^N$ *et* $t \in [0,T)$,

$$\left|\partial_t u(x,t) - |u|^{p-1} u(x,t)\right| \leq \epsilon \left|u(x,t)\right|^p + C.$$

Les estimations uniformes citées ci-dessus ont permis d'obtenir de nouveaux résultats d'explosion pour l'équation (4). On cite par exemple la stabilité du profil à l'explosion (voir Fermanian, Merle et Zaag [FKMZ00]) et la régularité de l'ensemble d'explosion (voir Zaag [Zaa06]).

Dans [MZ98a] et [MZ00], les auteurs montrent le Théorème de Liouville en variables auto-similaires. Ils utilisent la fonctionelle de Lyapunov suivante associée à (7) :

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \right) \rho dy, \text{ où } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \quad (8)$$

La preuve s'appuie fondamentalement sur la linéarisation de (7) autour de la solution stationnaire κ quand $s \to -\infty$. En s'inspirant du travail de Filippas et Kohn [FK92b], Merle et Zaag montrent qu'il y a au plus trois façons pour que $w \to \kappa$ quand $s \to -\infty$. Puis, ils montrent que l'une de ces façon correspond au cas $w(y,s) = \varphi(s - s_0)$ pour un certain $s_0 \in \mathbb{R}$ où $\varphi(s) = \kappa\,(1+e^s)^{-\frac{1}{p-1}}$. Dans les deux autres cas, les auteurs trouvent une contradiction en utilisant le critère d'explosion suivant :

Soit w une solution de (7) qui satisfait $I(w(s_0)) > 0$ pour un certain $s_0 \in \mathbb{R}$ où

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1} \left(\int_{\mathbb{R}^N} |w(y,s)|^2 \rho(y) dy \right)^{\frac{p+1}{2}}.$$

Alors, w explose en un certain temps $S > s_0$.

Dans notre travail [Nou08], on trouve que dans le cas positif (traité dans [MZ98a]), on peut éviter la partie longue et technique concernant la linéarisation autour de κ quand $s \to -\infty$. Plus précisement, on trouve que le Théorème de Liouville dans le cas positif peut être démontré grâce au critère d'explosion de Kaplan pour l'équation (4) déjà cité dans la page 5 (voir [Kap63]) et aux travaux de Giga et Kohn (voir [GK85], [GK87] et [GK89]). L'objet de notre travail est de présenter une démonstration plus simple que la démonstration de [MZ98a]. Notre démonstration a clairement un intérêt pédagogique. Bien sûr, pour des solutions sans signe on ne sait pas faire autrement que d'utiliser les techniques développées par [MZ00], qui est très proche du papier précédant [MZ98a].

2.2 Théorème de Liouville pour une équation de la chaleur sans structure du gradient et applications

Dans cette sous-section, nous présentons notre article [NZ08] écrit en collaboration avec H. Zaag, accepté pour publication dans les *Transactions of the American Mathematical Society*.

Dans ce travail, on suppose (5) et on s'intéresse à l'équation de la chaleur semi-linéaire à valeurs complexes suivante

$$\partial_t u = \Delta u + (1+i\delta)|u|^{p-1}u, \; u(0,x) = u_0(x), \quad (9)$$

où $u(t) : \mathbb{R}^N \to \mathbb{C}$ et $\delta \in \mathbb{R}$.
Notons que la non-linéarité dans cette équation n'est pas un gradient. Notons aussi que (9) est un cas particulier de l'équation complexe de Ginzburg-Landau

$$\partial_t u = (1+i\beta)\Delta u + (\epsilon + i\delta)|u|^{p-1}u - \gamma u, \text{ où } (x,t) \in \mathbb{R}^N \times (0,T), \quad (10)$$

β, δ et γ sont réels, $p > 1$ et $\epsilon = \pm 1$.
Notre ambition était de démontrer un Théorème de Liouville pour (9), à l'instar de celui démontré par Merle et Zaag dans [MZ98a] et [MZ00] pour $\delta = 0$. Cet objectif n'avait rien d'évident au premier abord, car la preuve de Merle et Zaag s'appuie de manière fondamentale sur l'existence de la fonctionelle de Lyapunov (8) et sur un critère d'explosion (voir page 9), qui n'a pas d'équivalent dans (9), car la non-linéarité n'est pas un gradient.

Mais d'abord, nous allons évoquer les résultats connus pour l'équation (9), en nous conformant aux deux problématiques présentées dans la Section 1.

- **Construction :** Citons le travail de Zaag [Zaa98] qui a construit une solution explosive stable pour (9) et en a donné le profil à l'explosion. Il y a aussi le travail de Popp, Stiller, Kuznetsov et Kramer [PKK98], qui utilisent une approche formelle pour trouver des solutions explosives. Plus récement, Masmoudi et Zaag [MZ08a] ont généralisé à l'équation (10) le résulat de [Zaa98]. Dans [MZ08a], les auteurs donnent une méthode constructive pour montrer l'existence de solutions explosives stables sous certaines conditions portant sur les paramètres.

- **Description :** À notre connaisance, notre papier [NZ08] est le premier qui s'intéresse à la description. Les autres papiers ([Zaa98], [PKK98] et [MZ08a]) décrivent des solutions particulières. Suivant l'approche par les Théorèmes de Liouville, on montre le résultat suivant (voir Théorème 2 page 36) :

Théorème 0.3. *(Théorème de Liouville pour l'équation (9)). Sous la condition (5), il existe $\delta_0 > 0$ et $M : [-\delta_0, \delta_0] \to (0, +\infty]$ avec $M(0) = +\infty$, $M(\delta) \to +\infty$ quand $\delta \to 0$ tels que :*

Si $|\delta| \le \delta_0$ et u est solution de (9) qui satisfait $u(x,t)(T-t)^{\frac{1}{p-1}} \in L^\infty(\mathbb{R}^N \times (-\infty, T), \mathbb{C})$ et $\|u(x,t)(T-t)^{\frac{1}{p-1}}\|_{L^\infty(\mathbb{R}^N\times(-\infty,T),\mathbb{C})} \le M(\delta)$, alors, $u \equiv 0$ ou il existe $T_0 \ge T$ et $\theta_0 \in \mathbb{R}$ tels que pour tout $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{-\frac{1+i\delta}{p-1}} e^{i\theta_0}$.

Remarque : Comme le cas $\delta = 0$ a été traité par Merle et Zaag [MZ98a] et [MZ00], le cas $\delta \neq 0$ peut paraître comme une perturbation purement technique du cas $\delta = 0$. En effet, même si les énoncés se ressemblent, la démonstration du Théorème de Liouville dans le cas $\delta \neq 0$ n'est pas la même que celle dans le cas $\delta = 0$. Ceci est dû au fait que notre équation n'a pas la structure de gradient et que le problème linéarisé n'est plus auto-adjoint. C'est pourquoi on développe de nouvelles méthodes, d'où l'originalité de notre travail.

Applications pour des solutions explosives de (9) de type I.
Comme on l'a déjà vu dans la littérature récente ([MZ00], [MM00] et [MZ08b]), les Théorèmes de Liouville ont d'importantes applications à l'explosion pour les solutions explosives de l'équation (9) dites de 'type **I**', c'est-à-dire les solution $u(t)$ qui vérifient

$$\forall t \in [0,T),\ \|u(t)\|_{L^\infty} \le M(T-t)^{-\frac{1}{p-1}},$$

où T est le temps d'explosion. En d'autres termes, le taux d'explosion est donnée par l'EDO associée $u'(t) = (1+i\delta)|u(t)|^{p-1}u(t)$.

On sait que la solution de (9) construite dans [Zaa98] est de type **I** (de même pour la solution de l'équation de Ginzburg-Landau (10) construite dans [MZ08a]). Cependant, nous sommes incapables de montrer que *toutes* les solutions explosives de (9) sont de type **I** ou non.

Notons que lorsque $\delta = 0$, Giga et Kohn [GK85] ainsi que Giga, Matsui et Sasayama [GMS04] montrent que toutes les solutions explosives sont de type **I**, pour p souscritique (condition (5)). Lorsque $\delta \neq 0$, les méthodes de [GK85] et [GMS04] ne s'appliquent pas car on n'a ni la positivité, ni la fonctionelle de Lyapunov.

Cependant, en procédant comme dans [MZ00] et [Zaa01], nous montrons les estimations suivantes pour des solutions explosives de l'équation (9) de type **I** : (voir Proposition 3 page 37)

Proposition 0.4. *(Estimations uniformes pour des solutions explosives de l'équation (9) de type* **I***) On suppose (5) et on considère $|\delta| \leq \delta_0$ et u une solution de (9) qui explose au temps T et vérifie :*

$$\forall t \in [0, T),\ \|u(t)\|_{L^\infty} \leq M(\delta)(T-t)^{-\frac{1}{p-1}},$$

où δ_0 et $M(\delta)$ sont définis dans le Théorème 0.3. Alors,

- *(i) (Estimations L^∞ pour les dérivées)*

$$\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \to \kappa \text{ et } \|\nabla^k u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}+\frac{k}{2}} \to 0$$
$$\text{quand } t \to T \text{ pour } k = 1,\ 2 \text{ ou } 3.$$

- *(ii) (Comportement uniforme comme une EDO) Pour tout $\varepsilon > 0$, il existe $C(\varepsilon)$ tel que pour tout $x \in \mathbb{R}^N$ et $t \in [\frac{T}{2}, T)$,*

$$\left|\partial_t u(x,t) - (1+i\delta)|u|^{p-1}u(x,t)\right| \leq \varepsilon |u(x,t)|^p + C.$$

2.3 Théorème de Liouville pour l'équation de la chaleur semi-linéaire en cas d'extinction

On présente dans ce travail les résultats de notre prépublication [Nou].
Ce papier concerne les solutions de l'équation de la chaleur semi-linéaire suivante :

$$\begin{cases} \partial_t u &= \partial^2_{xx} u - \dfrac{1}{u^\beta} \text{ dans } \mathbb{R} \times [0,T), \\ u(x,0) &= u_0(x) > 0 \text{ pour } x \in \mathbb{R}, \end{cases} \tag{11}$$

où $\beta \geq 3$ et

$$u_0, \quad \frac{1}{u_0} \in L^\infty(\mathbb{R}). \tag{12}$$

On dit que $u(t)$ s'éteint en temps fini T si u existe pour $t \in [0,T)$ et

$$\lim_{t\to T} \inf_{x\in\mathbb{R}} u(x,t) = 0. \tag{13}$$

Dans ce qui suit, on considère une solution u de (11) qui s'eteint en temps fini T. Un point a est dit point d'extinction s'il existe une suite $\{(a_n, t_n)\}$ telle que $a_n \to a$, $t_n \to T$ et $u(a_n, t_n) \to 0$ quand $n \to \infty$.
Les phénomènes d'extinction jouent un rôle important dans la physique des plasmas, la combustion, l'écologie et jouent aussi un rôle important dans la géométrie différentielle (voir par exemple Dziuk et Kawohl [DK91], Deng [Den92], Altschuler, Angenent et Giga [AAG95] ainsi que Galaktionov, Gerbi et Vázquez [GGV01]). L'étude du problème d'extinction (11) a commencé avec Kawarada [Kaw75]. Par la suite, plusieurs auteurs se sont intéressés aux questions d'existence et de comportement qualitatif des solutions qui s'éteignent en temps fini (voir Deng et Levine [DL89], Fila et Kawohl [FK92a], Fila, Kawohl et Levine [FKBL92], Levine [Lev93] et Dávila et Montenegro [DM05]). Il y a bien entendu beaucoup d'études numériques parmi lesquelles on notera celle de Liang, Lin et Tan [LLT07] qui compte parmi les plus récentes.
Intéressons nous d'abord à la littérature sur le sujet sous l'angle de la construction et la description, évoquées en Section 1.

- **construction :** Beaucoup d'auteurs ont trouvé des conditions suffisantes d'extinction dans (11). Nous choisissons de citer Merle et Zaag [MZ97] qui ont démontré l'existence d'une solution *stable*, ce qui a un intérêt physique certain.

- **Description :** La détermination du taux d'extinction quand $t \to T$ pour une solution u de (11) près d'un point d'extinction a mobilisé plusieurs auteurs comme Guo [Guo90], [Guo91a] et Filippas et Guo [FG93]. Pour des solution radiales dans $\mathbb{R}^N$ voir [Guo91b].

Théorème de Liouville

Ceci est le principal résultat de [Nou] (voir Théorème 4)

Théorème 0.5. *(Théorème de Liouville pour l'équation (11).) Soit u une solution positive et continue de (11) telle que pour tout $(x,t) \in \mathbb{R} \times (-\infty, T)$, $u(x,t) \geq M(T-t)^{\frac{1}{(\beta+1)}}$ et $|\partial_x u(x,t)| \leq M\,(T-t)^{\frac{1}{\beta+1}-\frac{1}{2}}$, pour un certain $M > 0$. Alors, il existe $T_0 \geq T$ tel que pour tout $(x,t) \in \mathbb{R} \times (-\infty, T)$, $u(x,T) = \kappa(T_0 - t)^{\frac{1}{(\beta+1)}}$.*

Applications à l'extinction

Notons que dans la littérature récente ([MZ00], [MM00] et [MZ08b]), les Théorèmes de Liouville ont une importante application à l'explosion. Nous ensons que dans le problème d'extinction il est possible d'obtenir des résultats similaires. Si on procède comme dans [MZ00] et [NZ08], on espère dans l'avenir proche démontrer les estimations suivantes :

- *(Comportement de type EDO) Soit $u(t)$ une solution positive de l'équation (11) qui s'éteint en temps fini $T > 0$. Alors, pour tout $\varepsilon > 0$, il existe $C_\varepsilon > 0$ tel que pour tout $t \in [\frac{T}{2}, T)$ et $x \in \mathbb{R}$,*

$$\left|\frac{\partial u}{\partial t} - u^{-\beta}\right| \leq \varepsilon |u|^{-\beta} + C_\varepsilon. \tag{14}$$

- (Bornes uniformes pour $u(t)$ au temps d'extinction) Soit $u(t)$ une solution de l'équation (11) qui s'éteint en temps fini T. On suppose aussi que $u_0'' - \frac{1}{u_0^\beta} \leq 0$. Alors,

$$\inf_{x\in\mathbb{R}} |u(x,t)| \to U_T(t) = (\beta+1)^{\frac{1}{\beta+1}}(T-t)^{\frac{1}{\beta+1}} \text{ quand } t \to T,$$

et

$$(T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}}\|\partial_x u(.,t)\|_{L^\infty(\mathbb{R})} + (T-t)^{-\frac{1}{\beta+1}+1}\|\partial_x^2 u(.,t)\|_{L^\infty(\mathbb{R})} \to 0 \text{ quand } t \to T.$$

3 Régularité de l'ensemble d'explosion pour une équation des ondes semi-linéaire

Dans ce chapitre, on présente les résultats de notre papier [Nou08a], accepté pour publication dans *Communications in Partial Differential Equations, 2008.*

Soit u une solution explosive et le graphe $x \mapsto T(x)$ son ensemble d'explosion. On s'intéresse à la régularité de l'ensemble d'explosion. Grâce à la vitesse de propagation finie, T est une fonction $1-$Lipschitzienne (voir Alinhac [Ali95]).
Dans [MZ08b], les auteurs démontrent un résultat général de régularité. Pour l'énoncer, nous avons besoin d'introduire la notion de point *non caractéristique.*
Un point $x_0 \in \mathbb{R}$ est dit *non caractéristique* s'il existe $\delta_0 = \delta_0(x_0) \in (0,1)$ et $t_0(x_0) < T(x_0)$ tels que

$$u \text{ est définie sur } \mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{15}$$

où

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \left\{(x,t)\big| t < \bar{t} - \bar{\delta}|x-\bar{x}|\right\}. \tag{16}$$

Dans la cas contraire le point est dit *caractéristique.*
On note $\mathcal{R}$ l'ensemble des points non caractéristiques. Il est clair que $\mathcal{R} \neq \emptyset$ (En effet si $x \mapsto T(x)$ atteint un minimum local en x_0 ; sinon, pour $|x|$ suffisement grand, $x \in \mathcal{R}$). Voici le résultat de [MZ08b] (voir Théorème 1 page 58) :

$\mathcal{R}$ est ouvert et T est de classe $\mathcal{C}^1$ dans $\mathcal{R}$.

Remarque : Caffarelli et Friedman [CF85] et [CF86] ont montré que $x \mapsto T(x)$ est une fonction $\mathcal{C}^1$ sous des conditions restrictives portant sur les données initiales.

Le résultat de [MZ08b] à été obtenu grâce à la détermination du profil à l'explosion en variables auto-similaires (voir [MZ07]) et à l'approche par Théorèmes de Liouville. En effet, dans [MZ08b], les auteurs montrent le résultat suivant :

Théorème 0.6. *(Théorème de Liouville pour l'équation (3)). Soit $u(x,t)$ une solution de*

(3) définie dans le cône $\mathcal{C}_{x^*,t^*,\delta^*}$ (16) telle que pour tout $t < T^*$,

$$
\begin{aligned}
&(T^*-t)^{\frac{2}{p-1}}\frac{\|u(t)\|_{L^2\left(B(0,\frac{T^*-t}{\delta^*})\right)}}{(T^*-t)^{1/2}}\\
&+(T^*-t)^{\frac{2}{p-1}+1}\left(\frac{\|u_t(t)\|_{L^2\left(B(0,\frac{T^*-t}{\delta^*})\right)}}{(T^*-t)^{1/2}}+\frac{\|\nabla u(t)\|_{L^2\left(B(0,\frac{T^*-t}{\delta^*})\right)}}{(T^*-t)^{1/2}}\right)\le C^*,
\end{aligned}
\tag{17}
$$

avec $(x^*,T^*)\in\mathbb{R}^2$, $\delta^*\in(0,1)$ et $C^*>0$. Alors, $u\equiv 0$ ou u peut être prolongée en une fonction (qu'on note aussi u), définie dans l'ensemble :

$$\{(x,t)\mid T_0+d_0(x-x^*)\}\supset \mathcal{C}_{x^*,t^*,\delta^*} \text{ par}$$

$$u(x,t)=\theta_0\kappa_0\frac{(1-d_0^2)^{\frac{1}{p-1}}}{(T_0-t+d_0(x-x^*))^{\frac{2}{p-1}}},$$

pour un certain $T_0\ge T^*$, $d_0\in[-\delta^*,\delta^*]$ et $\theta_0=\pm1$, avec

$$\kappa(d,y)=\kappa_0\frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ avec } \kappa_0=\left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}. \tag{18}$$

Dans [Nou08a], on améliore la régularité de $T(x)$ dans $\mathcal{R}$. Notre idée est reliée au travail de Zaag [Zaa02a], [Zaa02b] et [Zaa06] sur la régularité de l'ensemble d'explosion pour l'équation semi-linéaire de la chaleur (4).

Dans son travail, Zaag a utilisé l'idée qu'une meilleure description asymptotique de la solution près du point d'explosion induit des contraintes géométriques sur l'ensemble d'explosion résultant en une meilleure régularité.

Dans [Nou08a], on adapte cette idée à l'équation des ondes pour améliorer la régularité de $T(x)$. On obtient alors le résultat suivant :

Théorème 0.7. *Soit u une solution de (3) et $x\mapsto T(x)$ son ensemble d'explosion. Alors $T(x)$ est de classe $\mathcal{C}^{1,\mu_0}$ sur $\mathcal{R}$ pour un certain $\mu_0>0$.*

Remarque : On pense que $\mu_0\le 1$. Voir [Nou08a] pour une justification.

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Première partie

Étude de la formation de singularités en temps fini pour l'équation de la chaleur semi-linéaire

Chapitre 1

A simplified proof of a Liouville theorem for nonnegative solutions of a subcritical semilinear heat equation

In *Journal of Dynamics and Differential Equations* (to appear in 2008)

A simplified proof of a Liouville theorem for nonnegative solutions of a subcritical semilinear heat equations

Nejla Nouaili

We give a new proof of the Liouville theorem proved by Merle and Zaag for nonnegative solutions of the semilinear heat equation with power nonlinearity. Our proof has a pedagogical interest and is based on Kaplan's blow-up criterion.

Mathematical Subject classification : 35K05, 35K55, 35A20.
Keywords : heat equation, Liouville theorem, blow-up, Kaplan's criterion.

1.1 Introduction

In [MZ98a] and [MZ00], Merle and Zaag consider the following semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u \tag{1.1}$$

and prove the following Liouville theorem :

Theorem 1*(Merle-Zaag) Assume that*

$$p > 1 \text{ and } (N-2)p < N+2. \tag{1.2}$$

Consider u a solution of (1.1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$. Assume in addition that $|u(x,t)| \leq C(T-t)^{-\frac{1}{p-1}}$, for some constant $C > 0$. Then $u \equiv 0$ or there exists $T_0 \geq T$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \pm\kappa(T_0 - t)^{-\frac{1}{p-1}}$ with $\kappa = (p-1)^{-\frac{1}{p-1}}$.

Introducing the following *similarity variables* :

$$y = \frac{x}{\sqrt{T-t}},\ s = -\log(T-t),\ w(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t), \tag{1.3}$$

equation (1.1) is transformed in the following equation :

$$w_s = \Delta w - \frac{1}{2}y \cdot \nabla w - \frac{1}{p-1}w + |w|^{p-1}w, \tag{1.4}$$

and we get another equivalent formulation of the above Liouville theorem.

Theorem 1′*(Merle-Zaag) Assume (1.2) and consider $w(y,s)$ a bounded solution of (1.4), defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$. Then w is one of the following solutions : $w \equiv 0$, or $w = \pm\kappa$, or there exist $s_0 \in \mathbb{R}$, such that $w = \pm\varphi(s - s_0)$ with $\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}$.*

This theorem introduces a new approach in the study of equation (1.1), in the sense that it gives uniform estimates both in space and with respect to initial data. For instance, the following localization property is proved in [MZ98a] and [MZ00] :

Uniform ODE Behavior : Consider $T \le T_0$, $\|u_0\|_{C^2(\mathbb{R}^N)} \le C_0$ and $u(x,t)$ the solution of equation (1.1), with initial data u_0. Then, for all $\epsilon > 0$, there is $C(\epsilon, C_0, T_0)$ such that for all $x \in \mathbb{R}^N$ and $t \in [0,T)$,

$$\left|\frac{\partial u}{\partial t}(x,t) - |u|^{p-1}u(x,t)\right| \le \epsilon\, |u(x,t)|^p + C.$$

The above uniform estimate allowed to get new blow-up results for equation (1.1), unknown before, such as the stability of the blow up profile (see Fermanian, Merle and Zaag [FKMZ00]) and the regularity of the blow-up set (see Zaag [Zaa06]).
Moreover, the approach consisting in proving Liouville theorems in order to get new blow-up results has been successful for other parabolic equations with no gradient structure (see Nouaili and Zaag [NZ07]), hyperbolic equations like Korteweg de Vries (Martel and Merle [MM00]) and the wave equation (Merle and Zaag [MZ08a]).
In [MZ98a] and [MZ00], the authors prove the Liouville theorem in similarity variables. They use the following Lyapunov functional associated with (1.4) :

$$E(w) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1}\right)\rho dy, \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}. \tag{1.5}$$

The heart of the proof is the linearization of w (defined by (1.4)) around κ as $s \to -\infty$. Using similar ideas to Filippas and Kohn [FK92b], Merle and Zaag prove that there are at most three possible ways in which w goes to κ as $s \to -\infty$. Then, they show that one of the three cases corresponds to $w(y,s) = \varphi(s - s_0)$ for some $s_0 \in \mathbb{R}$ where $\varphi(s) = \kappa\,(1+e^s)^{-\frac{1}{p-1}}$. In the other two cases, they find a contradiction using the following blow-up criterion :

Let w be a solution of (1.4) which satisfies $I(w(s_0)) > 0$ for some $s_0 \in \mathbb{R}$ where

$$I(w(s)) = -2E(w(s)) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^N} |w(y,s)|^2\,\rho(y)dy\right)^{\frac{p+1}{2}}.$$

Then, w blows up at some time $S > s_0$.

In this note, we found that in the nonnegative case (treated in [MZ98a]), we could avoid the long and technical linearization around κ as $s \to -\infty$ and the application of the blow-up criterion. More precisely, we found that the Liouville theorem in the nonnegative case follows from Kaplan's blow-up criterion for equation (1.1) (see [Kap63]) and the work of Giga and Kohn (see [GK85], [GK87] and [GK89]), which are more simple ingredients. The aim of this note is to present this more simple proof, which is pedagogically easier then the analysis of [MZ98a]. Of course, for unsigned solutions, we cannot escape the proof given in [MZ00], which heavily relies on the preceding paper [MZ98a].

We proceed in two sections :
In section 2, for the reader's convenience, we give and prove our version of Kaplan's criterion, making our paper more self contained.
In section 3, we give our proof of the Liouville theorem in the nonnegative case.

Acknowledgment : We would like to thank the referee for his careful reading and helpful remarks. He suggests the use of Kaplan's blow-up criterion instead of Fujita's, which we used in the first version of this paper.

1.2 Kaplan's blow-up criterion

In the following, we give our version of Kaplan's criterion :

Proposition 1.2.1. *(Kaplan's blow-up criterion for equation (1.1)) There exists $M > 0$ such that if $V_0(x)$ in $L^\infty(\mathbb{R}^N)$ is nonnegative and satisfies*

$$\forall |x| \leq M, \quad V_0(x) \geq \frac{\kappa}{2},$$

then the solution $V(x,t)$ of equation (1.1) with initial data V_0 blows up in finite time $T > 0$.

Proof : Here we use Kaplan's method introduced in [Kap63] (see Theorem 8 page 327). Note that since $V_0(x)$ is nonnegative, the same holds for $V(x,t)$ whenever it exists. We note by $\lambda > 0$ the first eigenvalue of $-\Delta$ on the ball $B(0,M)$ and by $\psi(x)$ the corresponding eigenfunction to λ. In other words, we have

$$\begin{cases} -\Delta\psi &= \lambda\psi \text{ in } B(0,M), \\ \psi &= 0 \text{ on } \partial B(0,M). \end{cases}$$

Note that $\psi(x) \geq 0$ in $B(0,M)$ and from scaling arguments, we have $\lambda = \frac{\lambda_1}{M^2}$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on the ball $B(0,1)$. We also assume that $\int_{B(0,M)} \psi(x)dx = 1$. We define $\hat{V}(t) = \int_{B(0,M)} V(x,t)\psi(x)dx$. Multiplying both sides of (1.1) by $\psi(x)$ and integrating over $B(0,M)$, then using Jensen's inequality and integration by parts, we obtain

$$\begin{cases} \hat{V}'(t) &\geq -\lambda\hat{V} + \hat{V}(t)^p \text{ wherever } V \text{ is defined}, \\ \hat{V}(0) &\geq \frac{\kappa}{2}, \end{cases} \tag{1.6}$$

(see [Kap63] for details).
Now, we note by $\Phi(t)$ the solution of the ODE :

$$\begin{cases} \Phi'(t) = & \Phi(t)^p - \lambda\Phi(t), \\ \Phi(0) = & \frac{\kappa}{2}. \end{cases}$$

Since $\lambda = \frac{\lambda_1}{M^2}$, we can see that taking M large enough, $\Phi(t)$ blows up at some finite time $T_0 > 0$, hence by (1.6), $\hat{V}(t) \geq \Phi(t)$ and $\hat{V}$ blows up at some earlier time.
Using the fact that $\sup_{x\in\mathbb{R}^N} V(x,t) \geq \hat{V}(t)$, we conclude that $V(x,t)$ blows up in finite time $T > 0$. ■

1.3 Our new proof of the Liouville Theorem

We consider w a nonnegative, global and bounded solution of (1.4). We proceed in two steps. First, we find limits $w_{\pm\infty}$ for w as $s \to \pm\infty$ and reach a conclusion in some trivial cases. In a second step, we focus on the case where $w_{-\infty} = \kappa$ as $s \to -\infty$ and conclude the proof.

Step 1 : Limits of w as $s \to \pm\infty$

We recall respectively, Theorem 1 and Proposition 5 from [GK85] (page 305 and 309), under the condition (1.2).

Proposition 1.3.1. *(Stationary problem of (1.4) (Giga-Kohn)) The only global solutions in $L^\infty(\mathbb{R}^N)$ of*

$$0 = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w,$$

are the constant ones $w \equiv 0$, $w \equiv -\kappa$ and $w \equiv \kappa$.

Proposition 1.3.2. *(Limits of w as $s \to \pm\infty$ (Giga-Kohn)) Let w be a bounded global solution of (1.4) in $\mathbb{R}^N$. Then, $\lim_{s\to\infty} w(y,s)$ exists and equals 0 or $\pm\kappa$. The convergence takes place in $C^2(B(0,R))$ for any $R > 0$. The corresponding statements holds also for the limits as $s \to -\infty$.*

From the propositions above and the positivity of w, we have $w_{\pm\infty} \equiv 0$ or $w_{\pm\infty} \equiv \kappa$. Since E is a Lyapunov functional for w, one gets from (1.4) and (1.5) :

$$0 \leq \int_{-\infty}^{+\infty} ds \int_{\mathbb{R}^N} \left| \frac{\partial w}{\partial s}(y,s) \right|^2 \rho dy = E(w_{-\infty}) - E(w_{+\infty}). \tag{1.7}$$

Therefore, since $E(\kappa) > 0 = E(0)$, there are only two cases :

- $E(w_{-\infty}) = E(w_{+\infty})$. This implies that $\dfrac{\partial w}{\partial s} \equiv 0$, hence w is a stationary solution of (1.4) and $w \equiv 0$ or $w \equiv \kappa$ by Proposition 1.3.1.
- $E(w_{-\infty}) - E(w_{+\infty}) > 0$. This occurs only if $w_{+\infty} \equiv 0$ and $w_{-\infty} \equiv \kappa$. It remains to treat this case :

Step 2 : Case where $w \to \kappa$ as $s \to -\infty$

Consider $M > 0$ given by Proposition 1.2.1. From Proposition 1.3.2, there is some time s^* negative and large enough such that

$$w(y, s^*) \geq \frac{\kappa}{2} \text{ for all } |y| < M. \tag{1.8}$$

Introducing $v(x,t)$ defined by

$$v(x,t) = (1-t)^{-\frac{1}{p-1}} w(y, s+s^*) \text{ where } y = \frac{x}{\sqrt{1-t}} \text{ and } s = -\log(1-t), \tag{1.9}$$

we see that the function v is defined for all $x \in \mathbb{R}^N$ and $t < 1$, satisfies (1.1) and

$$\forall x \in \mathbb{R}^N, \quad v(x,0) = w(y, s^*). \tag{1.10}$$

If we consider $V(x,t)$ the solution of (1.1) with initial condition $V(x,0) = v(x,0)$, then from (1.8), (1.10) and Proposition 1.2.1, we have that V blows up at some finite time $T > 0$.
Since we have from uniqueness for (1.1) that

$$\forall (x,t) \in \mathbb{R}^N \times [0,1), \quad V(x,t) = v(x,t),$$

this gives $T \geq 1$. Extending $v(x,t)$ for $t \geq 1$ (if ever $T > 1$) by $v(x,t) = V(x,t)$, we see that $v(x,t)$ is a solution of (1.1) defined for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$ and which blows up at time $T \geq 1$ (note in particular that (1.9) still holds)

Now, if we consider $a \in \mathbb{R}^N$ a blow-up point of v and introduce the following *similarity variables* :

$$y' = \frac{x-a}{\sqrt{T-t}},\ s' = -\log(T-t),\ w_a(y',s') = (T-t)^{\frac{1}{p-1}} v(x,t), \tag{1.11}$$

then, we see from Giga and Kohn [GK87], Theorem 3.7 (page 17) that :

$$\forall s' \in \mathbb{R},\ \|w_a(s')\|_{L^\infty} \leq C_1, \text{ where } C_1 > 0, \tag{1.12}$$

and from [GK89], Corollary 3.4 (page 872), that :

$$w_a(y',s') \to \kappa \text{ as } s' \to +\infty, \text{ uniformly on compact sets.} \tag{1.13}$$

In the following, we are looking for the limit of w_a as $s' \to -\infty$. Using (1.9) and (1.11), we obtain :

$$\begin{aligned} &w_a(y',s') = (1-\sigma)^{-\frac{1}{p-1}} w(y,s) \ \text{ where } \sigma = (T-1)e^{s'}, \\ &y = \frac{y' + a e^{s'/2}}{\sqrt{1-\sigma}} \quad \textit{and} \quad s = s' - \log(1-\sigma) + s^*. \end{aligned} \tag{1.14}$$

Since $w(y,s) \to \kappa$ as $s \to -\infty$ uniformly on compact sets and $\sigma \to 0$ as $s' \to -\infty$, this gives that

$$w_a(y',s') \to \kappa \text{ as } s' \to -\infty, \text{ uniformly on compact sets.} \tag{1.15}$$

From parabolic regularity and the continuity of the energy $E(w_a)$, we get from (1.13) and (1.15)

$$E(w_a(s')) \to E(\kappa) \text{ as } s' \to \pm\infty.$$

Using the energy identity (1.7) for w_a, we conclude that

$$\int_{-\infty}^{+\infty} ds' \int_{\mathbb{R}^N} \left| \frac{\partial w_a}{\partial s}(y',s') \right|^2 \rho dy' = E(\kappa) - E(\kappa) = 0,$$

hence w_a is just a function of y'. Using the bound (1.12) and Proposition 1.3.1, we see that w_a is constant. Using limits (1.13) and (1.15), this yields $w_a \equiv \kappa$. Consequently, we obtain from (1.9), (1.11) and (1.14) :

$$w(y,s) = w_a(y',s')\left(1+(1-T)e^{s'}\right)^{1/(p-1)} = \kappa\left(1+(1-T)e^{s'}\right)^{1/(p-1)}.$$

Since we have from (1.14) $e^{s'} = \dfrac{e^{s-s^*}}{1+e^{s-s^*}(T-1)}$, we get

$$w(y,s) = \kappa\left(1+(T-1)e^{s-s^*}\right)^{-1/(p-1)},$$

hence $w(y,s) = \kappa$ if $T = 1$ or $w = \kappa\left(1+e^{s-s_0}\right)^{-1/(p-1)}$ if $T > 1$, where $s_0 = -\log(T-1)+s^*$, which is the desired conclusion. This ends the proof of the Liouville Theorem in the nonnegative case.

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Chapitre 2

A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up

In *Transactions of the American Mathematical Society* (to appear in 2008)

A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up

Nejla Nouaili and Hatem Zaag

We prove a Liouville Theorem for a vector valued semilinear heat equation with no gradient structure. Classical tools such as the maximum principle or energy techniques break down and have to be replaced by a new approach. We then derive from this theorem uniform estimates for blow-up solutions of that equation.

Mathematical Subject classification : 35B05, 35K05, 35K55, 74H35.
Keywords : Blow-up, Liouville theorem, uniform estimates, heat equation, vector-valued.

2.1 Introduction

This paper is concerned with blow-up solutions of the semilinear heat equation

$$\partial_t u = \Delta u + F(u), \tag{2.1}$$

where $u(t) : x \in \mathbb{R}^N \to \mathbb{R}^M$, Δ denotes the Laplacian and $F : \mathbb{R}^M \to \mathbb{R}^M$ is not necessarily a gradient. We say that $u(t)$ blows up in finite time T, if $u(t)$ exists for all $t \in [0, T)$ and

$$\lim_{t\to T} \|u(t)\|_{L^\infty} = +\infty.$$

We note that an extensive literature is devoted to the study of equation (2.1). Many results were found using monotonicity properties, maximal principle (valid for scalar equations) or energy techniques (valid when F is a gradient). See for example [Wei84], [Fuj66], [Bal77], [Lev73]. Unfortunately, there are important classes of PDEs where these techniques break down. For example, equations of the type (2.1), where F is not a gradient, or PDEs coming from geometric flows; see for example a review paper by Hamilton [Ham95].

In this work, we would like to develop new tools for a class of equations where classical tools do not work, in particular, vector valued equations with no gradient structure. More precisely, we will consider the following reaction-diffusion equation.

$$u_t = \Delta u + (1 + i\delta)|u|^{p-1}u,\ u(0, x) = u_0(x), \tag{2.2}$$

where $u(t) : \mathbb{R}^N \to \mathbb{C}$, $\delta \in \mathbb{R}$ and

$$p > 1 \text{ and } (N - 2)p < (N + 2). \tag{2.3}$$

Note that the nonlinearity in this equation is not a gradient. Note also that (2.2) is a particular case of the Complex Ginzburg-Landau equation

$$\partial_t u = (1 + i\beta)\Delta u + (\epsilon + i\delta)|u|^{p-1}u - \gamma u, \text{ where } (x, t) \in \mathbb{R}^N \times (0, T), \tag{2.4}$$

β, δ and γ real, $p > 1$ and $\epsilon = \pm 1$.
This equation is mostly famous when $\epsilon = -1$. It appears in the study of various physical problems (plasma physics, nonlinear optics). It is in particular used as an amplitude equation near the onset of instabilities in fluid mechanics (see for example Levermore and Oliver [LO96]). In this case, Plecháč and Šverák [PŠ01] used matching techniques and numerical simulations to give a strong evidence for the existence of blow-up solutions in the focusing case, namely $\beta\delta > 0$.
The case $\varepsilon = 1$ is less famous. To our knowledge, there is only the work of Popp, Stiller, Kuznetsov and Kramer [PKK98], who use a formal approach to find blow-up solutions. More recently, Masmoudi and Zaag [MZ08a] gave a constructive method to show the existence of a stable blow-up solution under some conditions for the parameters.
Let us present in the following the known results for equation (2.2) and most importantly the research directions and open problems. In the study of the blow-up phenomenon for equation (2.2), we believe that there are two important issues :

Construction of examples of blow-up solutions : In this approach, one has to construct *examples* of solutions that blow up in finite time. In particular, one has to find conditions on initial data and/or parameters of the equation to guarantee that the solution blows up in finite time. For equation (2.2), we recall the result obtained by Zaag [Zaa98] (the range of δ has been widened in [MZ08a]) :

For each $\delta \in (-\sqrt{p}, \sqrt{p})$,
i) equation (2.2) has a solution $u(x,t)$ on $\mathbb{R}^N \times [0,T)$ which blows up in finite time $T > 0$ at only one blow-up point $a \in \mathbb{R}^N$,
ii) moreover, we have

$$\lim_{t \to T} \|(T-t)^{\frac{1+i\delta}{p-1}} u(a + ((T-t)|\log(T-t)|)^{\frac{1}{2}} z, t) - f_\delta(z)\|_{L^\infty(\mathbb{R}^N)} = 0 \qquad (2.5)$$

with

$$f_\delta(z) = (p - 1 + \frac{(p-1)^2}{4(p-\delta^2)}|z|^2)^{-\frac{1+i\delta}{p-1}},$$

iii) there exists $u_* \in \mathcal{C}(\mathbb{R}^N \backslash \{a\}, \mathbb{C})$ such that $u(x,t) \to u_*(x)$ as $t \to T$ uniformly on compact subsets of $\mathbb{R}^N \backslash \{a\}$ and

$$u_*(x) \sim \left[\frac{8(p-\delta^2)|\log|x-a||}{(p-1)^2|x-a|^2}\right]^{\frac{1+i\delta}{p-1}} \quad \text{as } x \to a.$$

Remark : In [MZ08a], the same result was proved for equation (2.4), where the linearized operator around the expected profile is much more difficult to study.

Asymptotic behavior for any arbitrary blow-up solution : In this approach, one takes *any arbitrary* blow-up solution for equation (2.2) and tries to describe its blow-up behavior. More precisely, it consists in the determination of the asymptotic profile (that is a function from which, after a time dependent scaling, $u(t)$ approaches as $t \to T$) of the blow-up solution.

In earlier literature, the determination of the profile is done through the study of entire solutions (defined for all time and space) of the equation. See for example Grayson and Hamilton [GH96] for the case of the harmonic map heat flow and Giga and Kohn [GK85] for the heat equation (that is $\delta = 0$ in (2.2) ; there, the authors prove a Liouville Theorem which turns to be the trivial case of the Liouville Theorem proved by Merle and Zaag in [MZ98a] and [MZ00] and stated in Proposition 2.3.2 below). Let us remark that the use of *Liouville theorems* was successful for elliptic equations (see Gidas and Spruck [GS81a] and [GS81b]).
More recently, the characterization of entire solutions by means of *Liouville Theorems* allowed to obtain more than the blow-up profile, namely *uniform* estimates with respect to initial data and the singular point. See for the heat equation Merle and Zaag [MZ98b], [MZ98a], [MZ00], for the modified Korteweg de Vries equation Martel and Merle [MM00], for the nonlinear Schrödinger equation Merle and Raphael [MR04], [MR05] and for the wave equation Merle and Zaag [MZ08] and [MZ08].

The existence of a Lyapunov functional is traditionally a crucial tool in the proof of Liouville theorems, like for the heat equation [MZ00] or the wave equation [MZ08]. One wanders whether it is possible to prove a Liouville theorem for a system with no Lyapunov functional. The first attempt was done by Zaag [Zaa01] for the following system

$$\partial_t u = \Delta u + v^p,\ \partial_t v = \Delta v + u^q, \tag{2.6}$$

and its selfsimilar version

$$\begin{aligned} \partial_s \Phi &= \Delta \Phi - \tfrac{1}{2} y \cdot \nabla \Phi + \Psi^p - \left(\tfrac{p+1}{pq-1}\right) \Phi, \\ \partial_s \Psi &= \Delta \Psi - \tfrac{1}{2} y \cdot \nabla \Psi + \Phi^q - \left(\tfrac{q+1}{pq-1}\right) \Psi. \end{aligned} \tag{2.7}$$

This is the result of [Zaa01] :

Consider $p_0 > 1$ such that $(N-2)p_0 < N+2$ and $M > 0$. Then, there exists $\eta > 0$ such that if $|p - p_0| + |q - p_0| < \eta$, then for any nonnegative (Φ, Ψ) solution of (2.7) such that for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, $\Phi(y, s) + \Psi(y, s) \le M$, then either $(\Phi, \Psi) = (0, 0)$ or $(\Phi, \Psi) = (\Gamma, \gamma)$ or $(\Phi, \Psi) = \left(\Gamma\left(1 + e^{s-s_0}\right)^{-\frac{p+1}{pq-1}}, \gamma\left(1 + e^{s-s_0}\right)^{-\frac{q+1}{pq-1}}\right)$ for some $s_0 \in \mathbb{R}$, where (Γ, γ) is the only nontrivial constant solution of (2.7) defined by

$$\gamma^p = \Gamma\left(\frac{p+1}{pq-1}\right) \text{ and } \Gamma^q = \gamma\left(\frac{q+1}{pq-1}\right). \tag{2.8}$$

Before [Zaa01], Andreucci, Herrero and Velázquez addressed the same question in [AHV97] but could not determine explicitly the third case. In some sense, they just gave the limits as $s \to \pm\infty$ (for a statement, see the remark after Proposition 2.2.2 below). The characterization of that third case is far more difficult then the rest. The lack of a Lyapunov functional was overcome thanks to an infinite dimensional blow-up criterion.
Following system (2.6), it was interesting to address the case of equation (2.2) for $\delta \neq 0$.

Like for system (2.6), there is no Lyapunov functional. On the contrary, no blow-up criterion is available and the set of non zero stationary solutions for the selfsimilar version is a continuum (see Proposition 2.2.1 below). For these two reasons, new tools have to be found, which makes our paper meaningful.
Another reason for our work is the full Ginzburg-Landau model (2.4) with $\beta \neq 0$. That case has one more difficulty since the linearized operator in selfsimilar variable becomes non selfadjoint, as one can see from [MZ08a]. Thus, this paper is a fundamental step towards the proof of a Liouville theorem for the full Ginzburg-Landau model (2.4), which we believe to be an open problem of great importance.

2.1.1 A Liouville theorem for system (2.2)

Our aim in this paper is to prove a Liouville theorem for equation (2.2). In order to do so, we introduce for each $a \in \mathbb{R}^N$, the following selfsimilar transformation :

$$w_a(y,s) = (T-t)^{\frac{(1+i\delta)}{p-1}} u(x,t),\ y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t). \tag{2.9}$$

If u is a solution of (2.2), then the function $w = w_a$ satisfies for all $s \geq -\log T$ and $y \in \mathbb{R}^N$:

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{(1+i\delta)}{(p-1)} w + (1+i\delta)|w|^{(p-1)} w. \tag{2.10}$$

We introduce also the Hilbert space

$$L^2_\rho = \{g \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}),\ \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy < +\infty\} \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

If g depends only on the variable $y \in \mathbb{R}^N$, we use the notation

$$\|g\|^2_{L^2_\rho} = \int_{\mathbb{R}^N} |g(y)|^2 e^{-\frac{|y|^2}{4}} dy.$$

If g depends only on $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, we use the notation

$$\|g(.,s)\|^2_{L^2_\rho} = \int_{\mathbb{R}^N} |g(y,s)|^2 e^{-\frac{|y|^2}{4}} dy.$$

The main result of the paper is the following Liouville theorem which classifies certain entire solutions (i.e. solutions defined for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$) of (2.10) :

Theorem 1. *(A Liouville theorem for equation (2.10))Assuming (2.3), there exist $\delta_0 > 0$ and $M : [-\delta_0, \delta_0] \to (0, +\infty]$ with $M(0) = +\infty$, $M(\delta) \to +\infty$ as $\delta \to 0$ and the following property :*

If $|\delta| \leq \delta_0$ and $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})$ is a solution of (2.10) with $\|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})} \leq M(\delta)$, then, either $w \equiv 0$ or $w \equiv \kappa e^{i\theta_0}$ or $w = \varphi_\delta(s - s_0) e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where $\varphi_\delta(s) = \kappa(1+e^s)^{-\frac{(1+i\delta)}{(p-1)}}$ and $\kappa = (p-1)^{-\frac{1}{p-1}}$.

Going back to the original variables $u(x,t)$, we rewrite this Liouville theorem in the following :

Theorem 2. *(A Liouville theorem for equation (2.2))Assuming (2.3), there exist $\delta_0 > 0$ and $M : [-\delta_0, \delta_0] \to (0, +\infty]$ with $M(0) = +\infty$, $M(\delta) \to +\infty$ as $\delta \to 0$ and the following property :*

If $|\delta| \le \delta_0$ and u is a solution of (2.2) satisfying $u(x,t)(T-t)^{\frac{1}{p-1}} \in L^\infty(\mathbb{R}^N \times (-\infty, T), \mathbb{C})$ and $\|u(x,t)(T-t)^{\frac{1}{p-1}}\|_{L^\infty(\mathbb{R}^N \times (-\infty,T),\mathbb{C})} \le M(\delta)$, then, $u \equiv 0$ or there exists $T_0 \ge T$ and $\theta_0 \in \mathbb{R}$ such that for all $(x,t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{-\frac{1+i\delta}{p-1}} e^{i\theta_0}$.

Remark : This result has already been proved by Merle and Zaag [MZ98a] and [MZ00] (see also Nouaili [Nou08]) when $\delta = 0$. In that case, $M(0) = +\infty$, which means that any L^∞ entire solution of (2.10), *with no restriction on the size of its norm,* is trivial (i.e. independent of space).
When $\delta \neq 0$, this conclusion holds only for "small" L^∞ norm (i.e. bounded by $M(\delta)$). We suspect that we cannot take $M(\delta) = +\infty$. In other words, we suspect that equation (2.10) under the condition (2.3) has nontrivial solutions in L^∞ with a high norm. We think that such solutions can be constructed in the form $w(y,s) = w_0(y)e^{i\omega s}$ with high ω and high $\|w_0\|_{L^\infty}$, as Popp *et al.* did through formal arguments in page 96 in [PKK98] when $\delta \sim \pm 3$ (which is outside our range).

Remark : Since this result was already known from [MZ98a] and [MZ00] when $\delta = 0$, the case $\delta \neq 0$ may appear as a straightfoward interesting perturbation technique of the case $\delta = 0$. If this is clearly true for the statement, it is certainly not the case for the method and the techniques, mainly because the gradient structure breaks down and the linearized problem is no longer selfadjoint (see the beginning of Section 2 for more details). We have to invent new tools which are far from being a simple perturbation technique. This makes the main innovation of our work.

Remark : One may think that our result is completely standard in the context of dynamical systems. It happens that already in the case $\delta = 0$, standard methods such as the center manifold theory do not apply in our case as pointed by Filippas and Kohn in [FK92b] page 834-835. In particular, Proposition 2.3.5 page 51 below, whose statement is standard, does not follow from center manifold theory because the nonlinear term is not quadratic in the function space L^2_ρ.

2.1.2 Applications to type I blow-up solutions of (2.2)

As in previously cited blow-up recent literature ([MZ00], [MM00] and [MZ08]), Liouville theorems have important applications to blow-up for the so called 'type **I**' blow-up solutions of equation (2.2), that is, solutions satisfying

$$\forall t \in [0,T),\ \|u(t)\|_{L^\infty} \le M(T-t)^{-\frac{1}{p-1}},$$

where T is the blow-up time. In other words, the blow-up rate is given by the associated ODE $u' = (1+i\delta)|u|^{p-1}u$.

We know that the solution of (2.2) constructed in [Zaa98] is of type **I** (and the same holds for the solution of Ginzburg-Landau equation (2.4) constructed in [MZ08a]). However, we have been unable to prove whether *all* blow-up solutions of (2.2) are of type **I** or not. Note that when $\delta = 0$, Giga and Kohn [GK85] and Giga, Matsui and Sasayama [GMS04] prove that all blow-up solutions are of type **I**, provided that p is subcritical ($(N-2)p < N+2$). When $\delta \neq 0$, the methods of [GK85] and [GMS04] break down because we no longer have positivity or a Lyapunov functional.
However, following [MZ00] and [Zaa01], we can derive the following estimates for type **I** blow-up solutions of (2.2) :

Proposition 3. *(Uniform blow-up estimates for type* **I** *solutions) Assume (2.3), consider* $|\delta| \leq \delta_0$ *and a solution* u *of (2.2) that blows up at time* T *and satisfies*

$$\forall t \in [0,T),\ \|u(t)\|_{L^\infty} \leq M(\delta)(T-t)^{-\frac{1}{p-1}},$$

where δ_0 *and* $M(\delta)$ *are defined in Theorem 1. Then,*

– *(i) (L^∞ estimates for derivatives)*

$$\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \to \kappa \text{ and } \|\nabla^k u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}+\frac{k}{2}} \to 0$$
$$\text{as } t \to T \text{ for } k = 1,\ 2 \text{ or } 3.$$

– *(ii) (Uniform ODE Behavior) For all* $\varepsilon > 0$*, there is* $C(\varepsilon)$ *such that* $\forall x \in \mathbb{R}^N$*,* $\forall t \in [\frac{T}{2}, T)$*,*

$$\left|\frac{\partial u}{\partial t}(x,t) - (1+i\delta)|u|^{p-1}u(x,t)\right| \leq \varepsilon |u(x,t)|^p + C.$$

Remark : When $\delta = 0$, this result was already derived from the *Liouville theorem* in [MZ98a] and [MZ00]. It happens that adapting that proof to the case $\delta \neq 0$ is not straightforward, because the gradient structure is missing. However, unlike for the Liouville theorem, the adaptation is mainly technical. For the reader's convenience, we show in Section 4 how to adapt the proof of [MZ98a] and [MZ00] in the case $\delta \neq 0$.

Our paper is organized as follows : Section 2 and Section 3 are devoted to the proof of the Liouville theorem (we only prove Theorem 1 since Theorem 2 follows immediately from the selfsimilar transformation (2.9)). Note that Section 2 contains the main arguments with no details and Section 3 includes the whole proof with all the technical steps. Finally, we prove in Section 4 the applications to blow-up stated in Proposition 3.

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2.2 The main steps and ideas of the proof of the Liouville Theorem

In this section, we adopt a pedagogical point of view and explain the main steps and ideas of the proof with no technical details. These details are presented in Section 2.3. The

reader may think that our result is an interesting perturbation of the Liouville theorem proved in [MZ00] for $\delta = 0$. If this is true for the statement, it is certainly not the case for the proof for three structural reasons :
- the gradient structure breaks down when $\delta \neq 0$, which prevents us from using any energy method or blow-up criteria. To show blow-up, we need to find a very precise asymptotic behavior of the solution and show "by hand" that it cannot stay bounded.
- when $\delta \neq 0$, the linearized operator of (2.10) around the constant solution $w \equiv \kappa$ is no longer self adjoint and no general theory is applicable to derive eigenvalues directly. A careful decomposition of the solution is needed instead.
- since equation (2.2) is invariant under rotations in the complex plane ($u \to ue^{i\theta}$), this generates a null eigenvalue for the linear part of equation (2.10), and a precise modulation technique is needed, unlike the real valued case when $\delta = 0$.
The proof of the Liouville theorem is the same for $N = 1$ and $N \geq 2$ with subcritical p (see (2.3)). The only difference is in the multiplicity of the eigenvalues of the linearized operator of equation (2.10), which changes from 1 when $N = 1$ to a higher value when $N \geq 2$. In particular, one needs some extra notations and careful linear algebra in higher dimension. For the sake of clarity, we give here the proof when $N = 1$. The interested reader may find in section 4 (page 128) of [MZ00] how to get the higher dimensional case from the case $N = 1$. Clearly, the following statement is equivalent to Theorem 1 :

For any $M > 0$, there exists $\delta_0'(M) > 0$ such that for all $|\delta| \leq \delta_0'(M)$, if $w(y,s)$ is an entire solution of (2.10), defined for all $(y,s) \in \mathbb{R} \times \mathbb{R}$ and

$$\|w(.,s)\|_{L^\infty} \leq M, \tag{2.11}$$

then w depends only on the variable s.

In the following, we will prove this latter statement. Let us consider $M > 0$ and $w(y,s)$ satisfying (2.11), and prove that w is trivial provided that δ is small. As in [MZ00], the starting point is the investigation of the behavior of $w(y,s)$ as $s \to -\infty$.

Part 1 : Behavior of $w(y,s)$ as $s \to -\infty$.
In the case $\delta = 0$, the method of Giga and Kohn [GK85] proves that $w(y,s)$ approaches the set of stationary solutions of (2.10) $\{0, \kappa e^{i\theta} | \theta \in \mathbb{R}\}$ as $s \to -\infty$ in L^2_ρ. We would like to do the same here, that is why we give the stationary solutions of (2.10) in the following.

Proposition 2.2.1. *(L^∞ stationary solutions of (2.10)) Consider $\delta \neq 0$ and $v \in L^\infty(\mathbb{R}^N)$ a solution of*

$$0 = \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{1+i\delta}{p-1} v + (1+i\delta)|v|^{p-1}v. \tag{2.12}$$

Then, either $v \equiv 0$ or there exists $\theta_0 \in \mathbb{R}$ such that $v \equiv \kappa e^{i\theta_0}$.

Remark : When $\delta = 0$, the same result holds only for subcritical p verifying $(N-2)p \leq N+2$ and the proof due to Giga and Kohn is far from being trivial, see Theorem 1 (page 305) in [GK85].
Proof of Proposition 2.2.1 : Consider $v \in L^\infty(\mathbb{R}^N)$ a solution to (2.12). Multiplying (2.12) by $\overline{v}\rho$ and integrating over $\mathbb{R}^N$ gives after integration by parts

$$0 = -\int |\nabla v|^2 \rho - \frac{(1+i\delta)}{p-1} \int |v|^2 \rho + (1+i\delta) \int |v|^{p+1} \rho.$$

Since $\delta \neq 0$, identifying the imaginary and the real parts gives $\int |\nabla v|^2 \rho = 0$, hence $\nabla v \equiv 0$ and $\Delta v \equiv 0$. Plunging this in (2.12) yields the result.■
To prove that the solution approaches the set of stationary solutions, the method of Giga and Kohn breaks down, since it heavily relies on the existence of the following Lyapunov functional for equation (2.10) in the case $\delta = 0$:

$$E(w) = \frac{1}{2}\int |\nabla w|^2 \rho dy + \frac{1}{2(p-1)}\int |w|^2 \rho dy - \frac{1}{p+1}\int |w|^{p+1}\rho dy. \tag{2.13}$$

When $\delta \neq 0$, we don't have such a Lyapunov functional. Fortunately, a perturbation method used by Andreucci, Herrero and Velázquez, works here and yields the following :

Proposition 2.2.2. *For any $M > 0$, there exists $\delta_0'(M)$ such that if $|\delta| \leq \delta_0'$ and w is an arbitrary solution of (2.10) satisfying for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, $|w(y,s)| \leq M$, then either (i) $\|w(.,s)\|_{L^2_\rho} \to 0$ as $s \to -\infty$ or (ii) $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$.*

The next parts of the strategy (parts 2 and 3) investigate case (i) and (ii) of Proposition 2.2.2, which are certainly not of the same degree of difficulty.

Part 2 : Case where $w \to 0$ as $s \to -\infty$.
In this case, we have $w \equiv 0$. Rather than giving a proof, we simply explain here how the proof works. For the actual proof, we rely again on the method of Andreucci, Herrero and Velázquez (see Proposition 3.1, in section 3 of [AHV97]). Our argument is that the stationary solution of (2.10), which is identically zero, is stable in L^2_ρ, hence, no orbit can escape it, except the null orbit. To illustrate this, we write from equation (2.10) the following differential inequality for $h(s) \equiv \int_{\mathbb{R}} |w(y,s)|^2 \rho(y)dy$,

$$h'(s) \leq -\frac{2}{p-1}h(s) + 2\int_{\mathbb{R}} |w(y,s)|^{p+1}\rho(y)dy.$$

Using the regularizing effect of equation (2.10), we derive the following delay estimate (see Lemma 2.3.4 page 48 below) :

$$\forall s \in \mathbb{R},\ \int_{\mathbb{R}} |w(y,s)|^{p+1}\rho(dy) \leq C^* \left(\int_{\mathbb{R}} |w(y,s-s*)|^2 \rho(y)dy\right)^{\frac{p+1}{2}},$$

for some positive $s*$ and C^*. Therefore,

$$\forall s \in \mathbb{R},\ h'(s) \leq -\frac{2}{p-1}h(s) + C(M)h(s-s*)^{\frac{p+1}{2}}.$$

Using the fact that $h(s) \to 0$ as $s \to -\infty$ and delay ODE techniques, we show that $h(s)$ is driven by its linear part, hence for some $\varepsilon_0 > 0$ small enough, we have

$$\forall \sigma \in \mathbb{R},\ \forall s \geq \sigma + 1, h(s) \leq \varepsilon_0 e^{-\frac{2(s-\sigma)}{p-1}}.$$

Fixing $s \in \mathbb{R}$ and letting $\sigma \to -\infty$, we get that $h(s) \equiv 0$, hence $w \equiv 0$.

Now that case (i) of Proposition 2.2.2 has be handled, we consider case (ii) in the following.

Part 3 : Case where $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ **as** $s \to -\infty$**.**
The question to be asked here is the following : Does the solution converge to a particular $\kappa e^{i\theta_0}$ as $s \to -\infty$ or not ?
The key idea is to classify the L^2_ρ behavior of w as $s \to -\infty$. We proceed in 5 steps.

Step 1 : Formulation of the problem.
Note that the degree of freedom in case (ii) of Proposition 2.2.2 comes from the invariance of equation (2.2) under the rotation ($u \to ue^{i\theta}$). This invariance generates a zero mode for equation (2.10), which is difficult to control. The idea to gain this control and show the convergence of $w(y,s)$ is to use a modulation technique by introducing the following parametrization of the problem :

$$w(y,s) = e^{i\theta(s)}(v(y,s) + \kappa) \text{ with } \kappa = (p-1)^{-\frac{1}{(p-1)}}. \tag{2.14}$$

A natural choice would be to take $\theta(s)$ such that $\|w(.,s) - e^{i\theta(s)}\kappa\|_{L^2_\rho} = \inf_{\theta\in\mathbb{R}} \|w(y,s) - \kappa e^{i\theta}\|_{L^2_\rho}$. This is not our choice, we will instead choose $\theta(s)$ such that we kill the neutral mode mentioned above. More precisely, we claim the following :

Lemma 2.2.3. *There exists* $s_1 \in \mathbb{R}$ *and* $\theta \in C^1((-\infty, s_1], \mathbb{R})$ *such that*
(i) $\forall s \le s_1$, $\int(\Im(v) - \delta\Re(v))\rho = 0$, *where* v *is defined by (2.14).*
(ii) We have $\|v(.,s)\|_{L^2_\rho} \to 0$ *as* $s \to -\infty$.
(iii) For all $s \le s_1$, *we have*

$$|\theta'(s)| \le C\|v(.,s)\|^2_{L^2_\rho}. \tag{2.15}$$

With the change of variables (2.14), we focus in the following steps on the description of the asymptotic behavior of $v(y,s)$ and $\theta(s)$ as $s \to -\infty$. Using (2.14), we write the equation satisfied by $v(= v_1 + iv_2)$ as

$$\partial_s v = \tilde{\mathcal{L}}v - i\theta_s(v+\kappa) + G, \tag{2.16}$$

$$\text{where } G = (1+i\delta)\left\{|v+\kappa|^{p-1}(v+\kappa) - \kappa^p - \frac{v}{p-1} - v_1\right\}, \tag{2.17}$$

$$\text{satisfies } |G| \le C|v|^2 \text{ and } \left|G - (1+i\delta)\frac{1}{2\kappa}\left\{(p-2)v_1^2 + v_2^2 + 2v_1 v\right\}\right| \le C|v|^3. \tag{2.18}$$

A good understanding of our operator $\tilde{\mathcal{L}}v = \Delta v - \frac{1}{2}y\cdot\nabla v + (1+i\delta)v_1$ will be essential in our analysis. The following lemma provides us with the spectral properties of $\tilde{\mathcal{L}}$.

Lemma 2.2.4. *(Eigenvalues of* $\tilde{\mathcal{L}}$*).*
(i)$\tilde{\mathcal{L}}$ *is a* $\mathbb{R}-$*linear operator defined on* L^2_ρ *and its eigenvalues are given by*

$$\{1 - \frac{m}{2} | m \in \mathbb{N}\}.$$

Its eigenfunctions are given by $\{(1+i\delta)h_m, ih_m | m \in \mathbb{N}\}$ *where*

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!}(-1)^n y^{m-2n}. \tag{2.19}$$

We have : $\tilde{\mathcal{L}}((1+i\delta)h_m) = (1-\frac{m}{2})(1+i\delta)h_m$ *and* $\tilde{\mathcal{L}}(ih_m) = -\frac{m}{2}ih_m$.
(ii) Each $r \in L^2_\rho$ *can be uniquely written as*

$$r(y) = (1+i\delta)\tilde{r}_1(y) + i\tilde{r}_2(y) = (1+i\delta)\left(\sum_{m=0}^{+\infty}\tilde{r}_{1m}h_m(y)\right) + i\left(\sum_{m=0}^{+\infty}\tilde{r}_{2m}h_m(y)\right),$$

where :

$$\begin{array}{l} \tilde{r}_1(y) = \Re\{r(y)\} \text{ and } \tilde{r}_2(y) = \Im\{r(y)\} - \delta\Re\{r(y)\} \\ \text{and for} i = \{1,2\},\ \tilde{r}_{im} = \int \tilde{r}_i(y)\frac{h_m(y)}{\|h_m\|^2_{L^2_\rho}}\rho(y)dy. \end{array} \tag{2.20}$$

Remark : Note that the eigenvalues 1, 1/2 and 0 have a geometrical interpretation : they come from the invariance of equation (2.2) to translation in time ($\lambda = 1$) and space ($\lambda = 1/2$), dilations $u_\lambda(\xi,\tau) \to \lambda^{\frac{1}{p-1}}u(\xi\sqrt{\lambda},\tau\lambda)$ and multiplications by $e^{i\theta}$ (the group $\mathcal{S}^1$) for $\lambda = 0$.
Remark : Following (ii), we write each complex quantity (number or function) z as $z = z_1 + iz_2$ and $z = (1+i\delta)\tilde{z}_1 + i\tilde{z}_2$ with $z_{j=1,2}$, $\tilde{z}_{j=1,2} \in \mathbb{R}$. In particular, we write

$$\begin{array}{rcl} v(y,s) & = & (1+i\delta)\tilde{v}_1(y,s) + i\tilde{v}_2(y,s), \\ & = & (1+i\delta)\sum_{m=0}^{\infty}\tilde{v}_{1m}(s)h_m(y) + i\sum_{m=0}^{\infty}\tilde{v}_{2m}(s)h_m(y). \end{array} \tag{2.21}$$

Proof : Using the notation (2.21), we see that

$$\tilde{\mathcal{L}}v = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L}-\mathcal{I} \end{pmatrix}\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}, \tag{2.22}$$

where

$$\mathcal{L}h = \Delta h - \frac{1}{2}y\cdot\nabla h + h, \tag{2.23}$$

is a well-known self adjoint operator of $L^2_\rho(\mathbb{R},\mathbb{R})$ whose eigenfunctions are h_m (2.19), which are dilation of Hermite polynomials. Thus, the spectral properties of $\tilde{\mathcal{L}}$ directly derive from those of $\mathcal{L}$. The interested reader may find details in Lemma 2.2 page 590 from Zaag [Zaa98].■

Note from this Lemma that operator $\tilde{\mathcal{L}}$ has three nonnegative eigenvalues :
• $\lambda = 1$, with eigenfunction $(1+i\delta)h_0(y) = (1+i\delta)$.
• $\lambda = 1/2$, with eigenfunction $(1+i\delta)h_1(y) = (1+i\delta)y$.
• $\lambda = 0$, with two eigenfunctions $(1+i\delta)h_2(y) = (1+i\delta)(y^2-2)$ and $ih_0(y) = i$.
From (2.21) and (2.20), the coordinate of $v(y,s)$ along the direction ih_0 is

$$\begin{aligned} \tilde{v}_{20}(s) = & \int (\Im(v(y,s)) - \delta\Re(v(y,s)))\frac{h_0(y)}{\|h_0\|^2_{L^2_\rho}}\rho(y) \\ & = \int (\Im(v(y,s)) - \delta\Re(v(y,s)))\,\rho(y). \end{aligned}$$

Using (i) of Lemma 2.2.3, we see that the choice of $\theta(s)$ guarantees that

$$\forall s \le s_1,\ \tilde{v}_{20}(s) = 0. \tag{2.24}$$

In the next step, we will use the spectral information of $\tilde{\mathcal{L}}$ to derive the asymptotic behavior of v, then w as $s \to -\infty$.

Step 2 : Asymptotic behavior as $s \to -\infty$.
As $s \to -\infty$, we expect that the coordinates of v on the eigenfunctions for $\lambda \geq 0$ will dominate. These eigenfunctions are $(1+i\delta)$ when $\lambda = 1$, $(1+i\delta)y$ when $\lambda = 1/2$, $(1+i\delta)(y^2-2)$ or i when $\lambda = 0$. Note that for this latter case, the direction along i, already vanishes thanks to the choice of $\theta(s)$ (see (2.24)). So, if $\lambda = 0$ dominates, that is the coordinate of v on $(1+i\delta)(y^2-2)$ dominates, since the linear part vanishes, the equation is driven by the quadratic approximation $\dot{x} \sim -x^2$, that is $x \sim \frac{1}{s}$. Using (iii) of Lemma 2.2.3, we see that $\theta(s)$ has a limit as $s \to -\infty$, hence w converges from (2.14). More precisely, we have :

Proposition 2.2.5. *There exists $\theta_0 \in \mathbb{R}$ such that $\theta(s) \to \theta_0$ and $\|w(.,s) - \kappa e^{i\theta_0}\|_{L^2_\rho} \to 0$ as $s \to -\infty$. More precisely, one of the following situations occurs as $s \to -\infty$, for some $C_0 \in \mathbb{R}$ and $C_1 \in \mathbb{R}^*$,*

$$\begin{array}{lll} (i) & \|w(.,s) - \{\kappa + (1+i\delta)C_0 e^s\}e^{i\theta_0}\|_{L^2_\rho} & \leq Ce^{3s/2}, \\ (ii) & \|w(.,s) - \{\kappa + (1+i\delta)C_1 e^{s/2} y\}e^{i\theta_0}\|_{L^2_\rho} & \leq Ce^{(1-\varepsilon)s}. \\ (iii) & \|w(.,s) - e^{i\theta_0}\{\kappa + (1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2) - i\frac{(1+\delta^2)\delta\kappa^2}{4(p-\delta^2)^2}\frac{1}{s}\}\|_{L^2_\rho} & \leq C\frac{\log|s|}{s^2}, \end{array} \quad (2.25)$$

In Step 3, we show that case (i) yields the explicit solution $\varphi_\delta(s-s_0)$ for some s_0. In Steps 4 and 5, we rule out cases (ii) and (iii).
In comparison with the case $\delta = 0$, we can say that the difficulty in deriving Proposition 2.2.5 is only technical. One should bear in mind that the difficulty level is much lower than the obstacles we have in steps 4 and 5 to rule out cases (ii) and (iii) of Proposition 2.2.5.

Step 3 : Case where (i) holds.
Like for step 2, there is no real novelty in this step, the difficulty is purely technical. First we recall (i) from Proposition 2.2.5 :

$$\|w(.,s) - \{\kappa + (1+i\delta)C_0 e^s\}e^{i\theta_0}\|_{L^2_\rho} \leq Ce^{3s/2}. \quad (2.26)$$

Let us remark that we already have a solution $\hat{\varphi}(s)e^{i\theta_0}$ of (2.10) defined in $\mathbb{R} \times (-\infty, \hat{s}]$ for some $\hat{s} \in \mathbb{R}$ and which satisfies the same expansion as w :

$$\begin{array}{lll} \text{(a) if} \quad C_0 = 0, & \text{just take } \hat{\varphi} \equiv \kappa, & \\ \text{(b) if} \quad C_0 < 0, & \text{take } \hat{\varphi} \equiv \varphi_\delta(s-s_0), \text{ where } s_0 = -\log(-\frac{C_0(p-1)}{\kappa}), & \\ \text{(c) if} \quad C_0 > 0, & \text{take } \hat{\varphi} \equiv \varphi^*_\delta(s-s_0), \text{ where } s_0 = \log(\frac{C_0(p-1)}{\kappa}), & \\ \quad \text{and } \varphi^*_\delta(s) & = \kappa(1-e^s)^{-\frac{(1+i\delta)}{(p-1)}} & \end{array} \quad (2.27)$$

$\varphi^*_\delta(s)$ is a solution of (2.10) that blows up at $s = 0$, but is bounded for all $s \leq -1$. Note that, from (2.26) we have :

$$\|w(.,s) - \hat{\varphi}(s)e^{i\theta_0}\|_{L^2_\rho} \leq Ce^{3s/2}. \quad (2.28)$$

Since the difference between the two solutions of (2.10) is of order $e^{3s/2}$ and the largest eigenvalue of $\tilde{\mathcal{L}}$ is $1 < \frac{3}{2}$, this difference has to vanish leading to $w(y,s) = \hat{\varphi}(s)e^{i\theta_0}$ (remember that the largest eigenvalue matters, since $s \to -\infty$). Since case (c) violates the uniform bound (2.11), only cases (a) or (b) occur. More precisely, we have the following :

Proposition 2.2.6. *Assume that case (i) of Proposition 2.2.5 holds. Then, either* $w \equiv \kappa e^{i\theta_0}$ *or there exists* $s_0 \in \mathbb{R}$ *such that for all* $(y,s) \in \mathbb{R} \times \mathbb{R}$, $w(y,s) = \varphi_\delta(s - s_0)e^{i\theta_0}$ *for some* $\theta_0 \in \mathbb{R}$.

Steps 4 and 5 : Irrelevance of cases (ii) and (iii) of Proposition 2.2.5.

Step 4 and the following make the novelty of our work. Indeed, in the case $\delta = 0$ treated in [MZ00], cases (ii) and (iii) of Proposition 2.2.5 were ruled out thanks to a blow-up criterion based on energy methods. Indeed, when $\delta = 0$, Merle and Zaag used the Lyapunov functional for equation (2.10) introduced in (2.13). More precisely, they have the following blow-up criterion (see Proposition 2.1 page 111 in [MZ00]) :

Lemma 2.2.7. *(A blow-up criterion for equation (2.10) when* $\delta = 0$*). Let* W *be a solution of (2.10), (with* $\delta = 0$*), which satisfies :*

$$E(W(y,s_0)) < \frac{p-1}{2(p+1)} \left(\int_{\mathbb{R}^N} |W(y,s_0)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

for some $s_0 \in \mathbb{R}$*. Then,* W *blows-up at some time* $S > s_0$.

Still for $\delta = 0$, it is shown in [MZ00], when case (ii) or (iii) hold in Proposition 2.2.5, that

for some a_0 *and* s_0*, we have*

$$E(w_{a_0}(.,s_0)) < \frac{p-1}{2(p+1)} \left(\int w_{a_0}(y,s_0)^2 \rho(y) dy \right)^{\frac{p+1}{2}}, \tag{2.29}$$

where

$$w_{a_0}(y,s) = w(y + a_0 e^{\frac{s}{2}}, s), \tag{2.30}$$

is also a solution of (2.10).

A contradiction follows then since in the same time w_{a_0} is defined for all $s \in \mathbb{R}$ from (2.30) and has to blow-up by condition (2.29) and Lemma 2.2.7.

When $\delta \neq 0$, all this collapses. No perturbation method can allow us to use in any sense the Lyapunov functional or the blow-up criterion. We have to invent a new method to rule out cases (ii) and (iii) of Proposition 2.2.5. Let us explain our strategy only for case (iii), since it is quite similar for case (ii). From rotation invariance of equation (2.10), we assume that $\theta_0 = 0$.

Our source of inspiration is the study of (2.10), when $\delta = 0$ and $w \to \kappa$ as $s \to +\infty$ (and not $-\infty$) by Herrero and Velázquez [HV93] and Velázquez [Vel92], to obtain the (supposed to be generic) profile, starting with the following profile

$$w(y,s) = \kappa + \frac{\kappa}{2ps}(1 - \frac{1}{2}|y|^2) + o(\frac{1}{s}) \text{ as } s \to \infty.$$

The convergence here takes place in L^2_ρ and $L^\infty(|y| < R)$ for any $R > 0$.

Herrero and Velázquez extended this convergence to a larger set of the form $|y| \le K\sqrt{s}$, for any $K > 0$. They obtained :

$$\sup_{|y|<K\sqrt{s}} \left| w(y,s) - f_0(\frac{y}{\sqrt{s}}) \right| \to 0 \text{ as } s \to +\infty,$$

where $f_0 = \left((p-1) + \frac{(p-1)^2}{4p}\frac{|y|^2}{s}\right)^{-\frac{1}{p-1}}$ is a solution of

$$0 = \frac{1}{2}\xi \cdot \nabla f_0(\xi) - \frac{1}{p-1} f_0(\xi) + |f_0|^{p-1} f_0(\xi), \text{ where } \xi = \frac{y}{\sqrt{s}},$$

In some sense, we can say that f_0 is an approximate solution of (2.10) when $s \to \infty$, because

$$\|\partial_s f_0 - \left\{ \Delta f_0 + \frac{1}{2}\xi \cdot \nabla f_0 - \frac{1}{p-1} f_0 + |f_0|^{p-1} f_0 \right\} \|_{L^\infty} = \|\partial_s f_0 - \Delta f_0\|_{L^\infty} \le \frac{C}{s}.$$

We note that Velázquez's method is a kind of characteristic's method applied to the parabolic equation (2.10), where the Laplacian term is dropped down because the profile is flat. Here, we will use ideas from Velázquez to find the profiles of the solution in the variables $\frac{y}{\sqrt{-s}}$ ($ye^{s/2}$ in Step 5). We hope to find singular profiles, which violate the upper bound (2.11) on $w(y,s)$. Our candidate of the profile is $G\left(\frac{y}{\sqrt{-s}}\right)$, with $G(\xi) = \kappa\left(1 - \frac{(p-1)}{4(p-\delta^2)}\xi^2\right)^{-\frac{(1+i\delta)}{(p-1)}}$. In fact G is a solution of

$$0 = -\frac{1}{2}\xi \cdot \nabla G(\xi) - \frac{1+i\delta}{p-1} G + (1+i\delta)|G|^{p-1}G.$$

We can note (as in the case of f_0 defined below) that G is an approximate solution of (2.10) (for $|y| < K_0\sqrt{-s}$, where $K_0 = \sqrt{\frac{4(p-\delta^2)}{(p-1)}}$). We see also that G is singular for $|y| = K_0\sqrt{-s}$. Using Velázquez's technique to extend the convergence in (iii) of Proposition 2.2.5 from $|y| < R$ to larger regions $|y| < \varepsilon_0\sqrt{-s}$, with $\varepsilon_0 < K_0$, we can prove the following :

Proposition 2.2.8. *Assume that case (iii) from Proposition 2.2.5 holds, then there exists $\varepsilon_0 > 0$, such that :*

$$\begin{gathered}\lim_{s\to-\infty} \sup_{|y|\le\varepsilon_0\sqrt{-s}} \left| w(y,s) - G\left(\frac{y}{\sqrt{-s}}\right) \right| = 0, \\ \text{where } G(\xi) = \kappa\left(1 - \frac{(p-1)}{4(p-\delta^2)}\xi^2\right)^{-\frac{(1+i\delta)}{(p-1)}}.\end{gathered} \tag{2.31}$$

Imagine for a second that (2.31) holds for any arbitrary $\varepsilon_0 < K_0$. Since $|G(\xi)| \to \infty$ as $\xi \to K_0$, we can fix ε_0 large enough so that $|G(\varepsilon_0)| \geq 3M$. Taking $|s_0|$ large enough in (2.31), we then see that

$$|w(\varepsilon_0\sqrt{-s_0}, s_0)| \geq 2M,$$

which contradicts the upper bound (2.11). It happens that unlike the case $s \to \infty$, where $\xi = 0$ realizes the maximum of the profile f_0, here $\xi = 0$ realizes the minimum, which obliges us to take ε_0 small enough in order to use Velázquez' method of convergence extension. Since ε_0 is small in our approach, we remark from (2.31) that $w(y,s)$ is flat (i.e. close to a constant) in a large region, in the sense that

$$\sup_{\left|y-\frac{\varepsilon_0}{2}\sqrt{-s_0}|\leq 4|s_0\right|^{1/4}} \left|w(y,s_0) - G(\frac{\varepsilon_0}{2})\right| \to 0 \text{ as } s_0 \to -\infty.$$

Using a kind of continuity with respect to initial data for equation (2.10), we can show that for any $\varepsilon > 0$

$$\sup_{s_0\leq s\leq s_0^*-\eta} \left|w(\frac{\varepsilon_0}{2}\sqrt{-s_0}, s) - W_{\varepsilon_0}(s)\right| \to 0 \text{ as } s_0 \to -\infty, \tag{2.32}$$

where $s_0^* < +\infty$ is the lifespan of $W_{\varepsilon_0}(s)$ the space independent solution of (2.10), with $W_{\varepsilon_0}(s_0) \equiv G(\frac{\varepsilon_0}{2})$. It happens that W_{ε_0} can be computed explicitly :

$$W_{\varepsilon_0}(s) = \kappa\left(1 - e^{s-s_0}\frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)}\right)^{-\frac{(1+i\delta)}{p-1}}$$

and that it blows-up at time $s = s_0^* - \log\left(\frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)}\right) > s_0$, because ε_0 is small.
Taking $s_0^* = s_0 - \eta_0$, where $\eta_0 > 0$ is small enough such that $|W_{\varepsilon_0}(s_0^* - \eta_0)| \geq 3M$, we see from (2.32) that $|w(\frac{\varepsilon_0}{2}\sqrt{-s_0}, s_0^* - \eta_0)| \geq 2M$, which violates the upper bound (2.11).

Conclusion of Part 3 and the sketch of proof of the Liouville theorem : From Step 4 and 5 we see that cases (ii) and (iii) of Proposition 2.2.5 are ruled out. By Step 3, we obtain that $w \equiv \kappa e^{i\theta_0}$ or $w \equiv \varphi_\delta(s-s_0)e^{i\theta_0}$ for some real s_0 and θ_0, where φ_δ is defined in Theorem 1, which is the desired conclusion of Theorem 1. In Section 3, we give the details of the proof.

2.3 Details of the proof of the Liouville theorem

In this section, we give the whole proof of the Liouville theorem. We only prove Theorem 1 since Theorem 2 immediately follows though the selfsimilar transformation (2.9). Note that in Section 2, we already gave a sketch of the proof stressing only the main arguments. Thus this section is intended only to readers interested in technical details.
We adopt here the same sectioning as in Section 2 : three parts and Part 3 is Divided in five steps. Hence, we recommend that the reader reads first a given step in section 2

before reading the corresponding step in Section 3. As in Section 2, we prove Theorem 1 in its form given in the statement around (2.11). We consider $M > 0$ and a global solution $w(y, s)$ of (2.10), defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$ such that

$$\|w(y,s)\|_{L^\infty} \leq M.$$

Our goal is to find $\delta_0'(M) > 0$, such that if $|\delta| \leq \delta_0'(M)$, w depends only on the variable s. We proceed in three parts :
In Part 1, we show that when $s \to -\infty$, either $w \to 0$ or w approaches the set $\{\kappa e^{i\theta} | \theta \in \mathbb{R}\}$.
In Part 2, we handle the first case and show that $w \equiv 0$.
In Part 3, we linearize the equation around $\kappa e^{i\theta(s)}$, for some well chosen $\theta(s)$, and show that either $w \equiv \kappa e^{i\theta_0}$ or $w \equiv \varphi_\delta(s - s_0)e^{i\theta_0}$ for some real s_0 and θ_0, where φ_δ is defined in Theorem 1, which concludes the proof.
It happens that we rely on the analysis performed by Andreucci, Herrero and Velázquez [AHV97] for the system (2.7). That is the reason why we give Part 1 and Part 2 at once.

Parts 1 and 2 : Behavior of $w(y, s)$ as $s \to \infty$ and conclusion in the case where $w \to 0$ as $s \to -\infty$

In these parts, we investigate the behavior of w as $s \to -\infty$ and reach a conclusion in the easiest case. Following what we wrote in Part 1 of Section 2, we know from Proposition 2.2.1 that the set of stationary solutions of (2.10) consists in 0 and $\kappa e^{i\theta}$, where $\theta \in \mathbb{R}$. In order to prove that w approaches this set as $s \to -\infty$, we rely completely on the analysis performed in [AHV97] for the system (2.7). Indeed, no extra arguments is necessary for the present equation (2.10). That is why we only give the main arguments which make the proof of [AHV97] hold for equation (2.10) and refer the interested reader to [AHV97] for the details. Now, using the perturbation method of [AHV97], we have the following :

Proposition 2.3.1. *(A primary classification) For any $M > 0$, there exists $\delta_0'(M)$ such if $|\delta| \leq \delta_0'$ and w is an arbitrary solution to (2.10) satisfying for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $|w(y, s)| \leq M$, then, either (i)$(\|w(s)\| \equiv 0)$ or (ii)$\left(\inf_{\theta\in\mathbb{R}} \|w(s) - \kappa e^{i\theta}\| \to 0\right)$ as $s \to -\infty$.*

Remark : This result replaces Proposition 2.2.2 and Part 2 in section 2.
Remark : In [AHV97], the conclusion of the authors in Theorem 2 for system (2.7) is more accurate : either (Φ, Ψ) is $(0, 0)$ or (Γ, γ) defined in (2.8), or

$$(\Phi, \Psi) \to (\Gamma, \gamma) \text{ at } -\infty \text{ and } (\Phi, \Psi) \to (0, 0) \text{ at } +\infty.$$

Using the same technique for our equation (2.10), we get Proposition 2.3.1. Indeed, due to the fact that the set of non trivial stationary solutions is a continuum (see Proposition 2.2.1), we need a modulation technique to derive the case $w \equiv \kappa e^{i\theta}$, this case will be treated in Part 3.

Proof of Proposition 2.3.1 : This Proposition follows from the arguments developed for the twin system (2.7) in [AHV97], no more. To keep our paper in a reasonable length limit, we don't give the proof. However, we should mention the 3 fundamental features of (2.10) that one needs to check to be convinced that the proof of Andreucci, Herrero and Velázquez works here.

- Both systems are of parabolic type involving the same linear operator

$$\mathcal{L}_0 v = \frac{1}{\rho}\mathrm{div}(\rho \nabla v) = \Delta v - \frac{1}{2} y \cdot \nabla v,$$

and the zero solution is stable in both cases (for system (2.10), note that the linearized operator around the zero solution is

$$\mathcal{L}_0 w - \frac{(1+i\delta)}{p-1} w,$$

and its spectrum (on $\mathbb{C}$) is fully negative; it is given by $\{-\frac{m}{2} - \frac{(1+i\delta)}{p-1} | m \in \mathbb{N}\}$).
- When $p = q = p_0$ in (2.7), the authors give in (3.12) and Lemma 3.2 of [AHV97] a classification of entire solutions. In our case, when $\delta = 0$ in (2.10), we have the following Liouville theorem (see Theorem 1 in [MZ00])

Proposition 2.3.2. *(Merle-Zaag [MZ00]; A Liouville theorem for equation (2.10) with $\delta = 0$ and subcritical p) Assume (2.3) and let $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})$ be a solution of*

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w.$$

Then necessarily, one of the following cases occur :
a) $w \equiv 0$,
b)$\exists \theta \in \mathbb{R}$ such that $w(y,s) = \kappa e^{i\theta}$,
c)there exists $s_0 \in \mathbb{R}$, such that for all $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y,s) = \varphi(s - s_0) e^{i\theta_0}$ where $\theta_0 \in \mathbb{R}$ and

$$\varphi(s) = \kappa (1 + e^s)^{-\frac{1}{p-1}}.$$

Remark : Note that φ is the unique global solution (up to a translation) of

$$\varphi_s = -\frac{\varphi}{p-1} + \varphi^p,$$

satisfying $\varphi \to \kappa$ as $s \to -\infty$ and $\varphi \to 0$ as $s \to \infty$. The method of Andreucci, Herrero and Velázquez in [AHV97] is in fact a perturbation method around this result.
- The property of equation (2.2) saying "small L^2_ρ norm implies no blow-up locally" (note that this property replaces the Giga-Kohn property "small local energy implies no blow-up locally", which breaks down because we no longer have a gradient structure). This is the property :

Proposition 2.3.3. *For all $M > 0$, there exist positive η_0, C_0 and M_0 such that if $|\delta| \le 1$ and v is a solution of (2.2) satisfying*

$$\forall t \in [0,1), \quad \|v(t)\|_{L^\infty} \le M(1-t)^{-\frac{1}{p-1}} \text{ and if } \forall |x_0| \le 1,\ \|w_{x_0}(.,0)\|_{L^2_\rho} \le \eta, \quad (2.33)$$

for some $0 < \eta \le \eta_0$, where

$$y = \frac{x - x_0}{\sqrt{1-t}},\ s = -\log(1-t),\ w_{x_0}(y,s) = (1-t)^{\frac{1+i\delta}{p-1}} v(x,t),$$

then :
(i) For all $|x_0| \leq 1$ *and* $s \in [0, +\infty)$*,*

$$\|w_{x_0}(.,s)\|_{L^2_\rho} \leq C_0 \eta e^{-\frac{s}{p-1}}. \tag{2.34}$$

(ii) For all $|x| \leq 1$ *and* $t \in [0,1)$*, we have* $|v(x,t)| \leq M_0$*.*
Now, we write the following Lemma which will be useful in the proof of the proposition above.
Lemma 2.3.4. *(Regularizing effect of the operator* $\mathcal{L}$*) Assume that* $\psi(y,s)$ *satisfies*

$$\forall s \in [a,b], \quad \forall y \in \mathbb{R}, \quad \psi_s \leq (\mathcal{L} + \sigma)\psi \text{ and } 0 \leq \psi(y,s),$$

for some $a \leq b$ *and* $\sigma \in \mathbb{R}$*, where*

$$\mathcal{L}\psi = \Delta\psi - \frac{1}{2} y \cdot \nabla\psi + \psi = \frac{1}{\rho}\text{div}(\rho\nabla\psi) + \psi. \tag{2.35}$$

Then for any $r > 1$*, there exists* $C^* = C^*(r,\sigma) > 0$ *and* $s^* = s^*(r) > 0$ *such that*

$$\forall s \in [a + s^*, b], \ \left(\int_{\mathbb{R}} |\psi(y,s)|^r \rho(y) dy \right)^{1/r} \leq C^* \|\psi(., s - s^*)\|_{L^2_\rho}. \tag{2.36}$$

Proof : See Lemma 2.3 in [Vel93].■

Proof of Proposition 2.3.3 :
Consider $M > 0$, $|\delta| \leq 1$ and a solution v of (2.2) such that (2.33) holds for some $\eta > 0$, $|x_0| \leq 1$.
(i) For simplicity, we write w instead of w_{x_0}. Since w is a solution of (2.10), we multiply (2.10) by $w\rho$ and integrate to get

$$I'(s) \leq -\frac{2}{p-1} I(s) + \int |w(y,s)|^{p+1} \rho(y) dy, \text{ where } I(s) = \int |w(y,s)|^2 \rho(y) dy. \tag{2.37}$$

If we note $w = w_1 + iw_2$, then using Kato's inequality ($\Delta w_i \cdot sgn(w_i) \leq \Delta |w_i|$ with $i = 1, 2$) and the fact that w is bounded, we obtain by equation (2.10)

$$\partial_s(|w_1| + |w_2|) \leq \Delta(|w_1| + |w_2|) - \frac{y}{2} \cdot \nabla(|w_1| + |w_2|) + C(|w_1| + |w_2|),$$

for some $C = C(M) > 0$.
Using Lemma 2.3.4, we see that there exist $C^*(M) > 0$ and $s^* = s^*(p+1) > 0$ such that for all $s \geq s^*$

$$\int |w(y,s)|^{p+1} \rho(y) dy \leq C^* \|w(., s - s^*)\|^{p+1}_{L^2_\rho} \tag{2.38}$$

Now, we Divide the proof in two steps :
Step 1 : $0 \leq s \leq s^*$. Using (2.37) and the fact that w is bounded by $M > 0$ (see (2.33)), we get

$$I'(s) \leq \lambda I(s) \text{ for some } \lambda = \lambda(M) > 0,$$

hence $I(s) \le e^{\lambda s} I(0) \le e^{\lambda s}\eta^2 \le \frac{C_0^2}{2}\eta^2 e^{-\frac{2s}{p-1}}$, where we define $C_0^2 = 2e^{(\lambda+\frac{2}{p-1})s^*}$. This gives (2.34) for $0 \le s \le s^*$.
Step 2 : $s \ge s^*$. In this step, we argue by contradiction to prove (2.34) for all $s \ge s^*$. We suppose that there exists $s_1 > s^*$, such that

$$I(s) < (C_0\eta)^2 e^{-\frac{2s}{p-1}}, \text{ for all } s^* \le s < s_1 \tag{2.39}$$

$$I(s_1) = (C_0\eta)^2 e^{-\frac{2s_1}{p-1}}. \tag{2.40}$$

Let $F(s) = I(s)(C_0\eta)^{-2} e^{\frac{2s}{p-1}}$. From (2.37), (2.38), (2.39) and Step 1, we have for all $s^* \le s \le s_1$,

$$\begin{aligned} F'(s) &\le C^*(C_0\eta)^{-2} e^{\frac{2s}{p-1}} I(s-s^*)^{\frac{p+1}{2}} \\ &\le C^*(C_0\eta)^{p-1} e^{\frac{2s}{p-1}} e^{-(s-s^*)\frac{p+1}{p-1}} \le C^*(C_0\eta)^{p-1} e^{\frac{p+1}{p-1}s^*} e^{-s}. \end{aligned}$$

Since $F(s^*) \le \frac{1}{2}$ from the step above, we integrate the last inequality to obtain

$$\begin{aligned} F(s_1) &\le C^*(C\eta)^{p-1} e^{\frac{p+1}{p-1}s^*}\left(e^{-s^*} - e^{-s_1}\right) + F(s^*), \\ &\le C^*(C\eta)^{p-1} e^{\frac{2s^*}{p-1}} + \frac{1}{2} \le \frac{3}{4}, \end{aligned}$$

for $\eta \le \eta_0(M)$ small enough. This contradicts (2.40). Therefore, (2.34) holds
(ii) Applying parabolic regularity to equation (2.10) and using estimate (2.33), we get for all $|x_0| \le 1$, $R > 0$ and $|y| < R$, $|w_{x_0}(y,s)| \le M_0 e^{-\frac{s}{p-1}}$, hence for all $t \in [0,1)$, $|v(x_0,t)| \le M_0$, for some $M_0 = M_0(M)$. This ends the proof of Proposition 2.3.3.■

Part 3 : Case where $\inf_{\theta\in\mathbb{R}} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$.

We study case (ii) of Proposition 2.3.1. As we wrote in Part 3 of section 2, the natural question is to know whether w converges to a particular $\kappa e^{i\theta_0}$ as $s \to -\infty$ or not. A modulation technique will be essential to classify the L^2_ρ behavior for w and prove the convergence. We proceed in five steps.

- Step 1 is intended to the modulation technique.
- In Step 2, we show that the linearized problem of (2.10) around $\kappa e^{i\theta}$ has 3 nonnegative directions as $s \to -\infty$ ($\lambda = 1$, $1/2$ or 0), and that the component along one direction dominates the others. This gives a kind of profile for w with a uniform convergence on every compact sets.
- In Step 3, we show that the case where $\lambda = 1$ dominates corresponds either to $w = \kappa e^{i\theta_0}$ or $w = \varphi_\delta(s-s_0)e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where φ_δ is defined in Theorem 1.
- Steps 4 and 5 : To rule out cases where the directions $\lambda = 1/2$ or $\lambda = 0$ dominates, we use a geometrical method where the key idea is Velázquez's work [Vel92] to extend the convergence from compact sets to larger zones, where the profile appears to be singular, which violates the uniform bound (2.11) in w. These steps make the innovation of our work.

Step 1 : Formulation of the problem
Let us recall Lemma 2.2.3 from Section 2.

Lemma 2.2.3 *There exists* $s_1 \in \mathbb{R}$ *and* $\theta \in C^1((-\infty, s_1], \mathbb{R})$ *such that*
(i) $\forall s \le s_1$, $\int (\Im(v) - \delta \Re(v))\rho = 0$, *where* v *is defined by*

$$w(y,s) = e^{i\theta(s)}(v(y,s) + \kappa) \text{ with } \kappa = (p-1)^{-\frac{1}{(p-1)}}. \tag{2.41}$$

(ii) We have $\|v(.,s)\|_{L^2_\rho} \to 0$ *as* $s \to -\infty$.
(iii) For all $s \le s_1$, *we have*

$$|\theta'(s)| \le C\|v(.,s)\|^2_{L^2_\rho}. \tag{2.42}$$

Proof of Lemma 2.2.3 :
(i) Since $\inf_{\theta \in [0,2\pi]} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0$ as $s \to -\infty$ and $\|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho}$ is continuous as a function of θ and w, there exists $\tilde{\theta}(s)$ such that

$$\|g\|^2_{L^2_\rho} = \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy.$$

$$\|w(.,s) - \kappa e^{i\tilde{\theta}(s)}\|_{L^2_\rho} = \inf_{\theta \in [0,2\pi]} \|w(.,s) - \kappa e^{i\theta}\|_{L^2_\rho} \to 0 \text{ as } s \to -\infty. \tag{2.43}$$

we will slightly modify $\tilde{\theta}(s)$, so that if we define $v(y,s)$ by (2.41) for some $\theta(s)$ close to $\tilde{\theta}(s)$, then we have (i) of Lemma 2.2.3. We apply the implicit function theorem to $F : L^2_\rho \times \mathbb{R} \to \mathbb{R}$ defined by $F(w,\theta) = \int \left(\Im\left(we^{-i\theta} - \kappa\right) - \delta\Re\left(we^{-i\theta} - \kappa\right)\right)\rho$. Since we have $F\left(\kappa e^{i\tilde{\theta}}, \tilde{\theta}\right) = 0$ and $\frac{\partial F}{\partial \theta} = -\int \left(\Re\left(we^{-i\theta}\right) + \delta\Im\left(we^{-i\theta}\right)\rho\right)$, hence $\frac{\partial F}{\partial \theta}\left(\kappa e^{i\tilde{\theta}}, \tilde{\theta}\right) = -\kappa \ne 0$, using the implicit function theorem and (2.43), we obtain the existence and uniqueness of C^1 $\theta(w)$ such that $F(w, \theta(w)) = 0$ and $|e^{i\theta(w)} - e^{i\tilde{\theta}}| \le C_0\|w(.,s) - \kappa e^{i\tilde{\theta}}\|_{L^2_\rho}$.
(ii) Since, we have from (2.41)

$$\begin{aligned}\|v(.,)\|_{L^2_\rho} = \|w(.,s) - \kappa e^{i\theta(s)}\|_{L^2_\rho} &\le \|w(.,s) - \kappa e^{i\tilde{\theta}}\|_{L^2_\rho} + \kappa|e^{i\theta} - e^{i\tilde{\theta}}| \\ &\le (1 + C_0\kappa)\|w(.,s) - \kappa e^{i\tilde{\theta}(s)}\|_{L^2_\rho},\end{aligned}$$

using (2.43), we conclude that $\|v(.,s)\|_{L^2_\rho} \to 0$ as $s \to -\infty$.
(iii) writing $v = (1 + i\delta)\tilde{v}_1 + i\tilde{v}_2$, we rewrite (2.16) as follows

$$\tilde{v}_{1s} = \mathcal{L}\tilde{v}_1 + \theta'(s)(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1 \tag{2.44}$$

$$\tilde{v}_{2s} = (\mathcal{L} - 1)\tilde{v}_2 - \theta'(s)((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2 \tag{2.45}$$

where $\mathcal{L}$ is given in (2.35),

$$\tilde{G}_1 = \frac{p - \delta^2}{2\kappa}\tilde{v}_1^2 + \frac{1}{2\kappa}\tilde{v}_2^2 + O(|v|^3) \tag{2.46}$$

$$\tilde{G}_2 = (1+\delta^2)\frac{\tilde{v}_1(\delta\tilde{v}_1 + \tilde{v}_2)}{\kappa} + O(|v|^3). \tag{2.47}$$

Note that $(1 + i\delta)\tilde{G}_1 + i\tilde{G}_2 = G$ is defined in (2.17).
Now, we multiply (2.45) by ρ and integrate over $\mathbb{R}$ to get

$$\int \tilde{v}_{2s}\rho = \int \operatorname{div}(\rho\nabla\tilde{v}_2) - \int \theta'(s)((1+\delta^2)\tilde{v}_1 + \delta\tilde{v} + \kappa)\rho + \int \tilde{G}_2\rho.$$

From (2.20), we have $\tilde{v}_2 = \Im(v) - \delta\Re(v)$, we get from (i) of Lemma 2.2.3 $\int \tilde{v}_{2s}\rho = 0$. Since $\int \mathrm{div}(\rho\nabla\tilde{v}_2) = 0$ we obtain

$$\theta'(s)\int((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho = \int \tilde{G}_2\rho. \tag{2.48}$$

Using (2.47), we have

$$\left|\int \tilde{G}_2\rho\right| \leq C\int |v|^2\rho. \tag{2.49}$$

Recalling from (ii) that $\lim_{s\to-\infty}\|v\| = 0$, we have

$$\int((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho \to \int \kappa\rho \text{ as } s \to -\infty.$$

Thus, the conclusion follows from (2.48) and (2.49).■

Step 2 : Asymptotic behavior of v as $s \to -\infty$.
First, we recall the decomposition (2.21) :

$$\begin{aligned} v(y,s) &= (1+i\delta)\tilde{v}_1(y,s) + i\tilde{v}_2(y,s), \\ &= (1+i\delta)\textstyle\sum_{m=0}^{\infty}\tilde{v}_{1m}(s)h_m(y) + i\sum_{m=0}^{\infty}\tilde{v}_{2m}(s)h_m(y), \end{aligned}$$

and introduce

$$v_-(y,s) = (1+i\delta)\sum_{m=3}^{\infty}\tilde{v}_{1m}(s)h_m(y) + i\sum_{m=1}^{\infty}\tilde{v}_{2m}(s)h_m(y).$$

As we saw in Step 2 of Section 2, the modulation techniques gives $\tilde{v}_{20}(s) = 0$. Therefore, we have

$$v(y,s) = (1+i\delta)(\tilde{v}_{10}(s)h_0(y) + \tilde{v}_{11}(s)h_1(y) + \tilde{v}_{12}(s)h_2(y)) + v_-(y,s)$$

Using ODE techniques, we are able to prove the following :

Proposition 2.3.5. *(Classification of the behavior of $v(y,s)$ as $s \to -\infty$) As $s \to -\infty$, one of the following situations occurs :*

(i) $|\tilde{v}_{11}(s)| + |\tilde{v}_{12}(s)| + \|v_-(.,s)\|_{L^2_\rho} = o(\tilde{v}_{10}(s))$,
$\|v(.,s) - (1+i\delta)C_0e^s\|_{L^2_\rho} = O(e^{3s/2})$ *and* $|\theta'(s)| = O(e^{2s})$ *for some* $C_0 \in \mathbb{R}$.

(ii) $|\tilde{v}_{10}(s)| + |\tilde{v}_{12}(s)| + \|v_-(.,s)\|_{L^2_\rho} = o(\tilde{v}_{11}(s))$,
$\|v(.,s) - (1+i\delta)C_1e^{s/2}y\|_{L^2_\rho} = O(e^{(1-\varepsilon)s})$ *and* $|\theta'(s)| = O(e^s)$ *for some* $C_1 \in \mathbb{R}\setminus 0$ *and* $\varepsilon > 0$.

(iii) $|\tilde{v}_{10}(s)| + |\tilde{v}_{11}(s)| + \|v_-(.,s)\|_{L^2_\rho} = o(\tilde{v}_{12}(s))$,
$\|v(.,s) + (1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2)\|_{L^2_\rho} = O(\frac{\log|s|}{s^2})$ *and* $\theta'(s) = \frac{(1+\delta^2)\delta\kappa}{4(p-\delta^2)^2}\frac{1}{s^2} + O(\frac{\log|s|}{s^3})$.

Proof : As already pointed out by Filippas and Kohn in page 834-835 in [FK92b] in the case $\delta = 0$, we can't use center manifold theory to get the result. In some sense, we are not able to say that the nonlinear terms in (2.44) and (2.45) are quadratic in the function space L^2_ρ. However, using ODE techniques similar to those of [MZ98a] and [FM95], we manage to conclude. Since we add no real novelty, we leave the proof to Appendix 2.5.1. ■

Now, we recall Proposition 2.2.5 as it is a direct consequence of the Proposition above.
Proposition 2.2.5 *There exists $\theta_0 \in \mathbb{R}$ such that $\theta(s) \to \theta_0$ and $\|w(.,s) - \kappa e^{i\theta_0}\|_{L^2_\rho} \to 0$ as $s \to -\infty$. More precisely, one of the following situations occurs as $s \to -\infty$, for some $C_0 \in \mathbb{R}$ and $C_1 \in \mathbb{R}^*$,*

$$\begin{array}{llll} (i) & \|w(.,s) - \{\kappa + (1+i\delta)C_0 e^s\}e^{i\theta_0}\|_{L^2_\rho} & \leq & Ce^{3s/2}, \\ (ii) & \|w(.,s) - e^{i\theta_0}\{\kappa + (1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2) - i\frac{(1+\delta^2)\delta\kappa^2}{2(p-\delta^2)^2}\frac{1}{s}\}\|_{L^2_\rho} & \leq & C\frac{\log|s|}{s^2}, \\ (iii) & \|w(.,s) - \{\kappa + (1+i\delta)C_1 e^{s/2} y\}e^{i\theta_0}\|_{L^2_\rho} & \leq & Ce^{(1-\varepsilon)s}. \end{array} \quad (2.50)$$

Proof of Proposition 2.2.5 : From Proposition 2.3.5, we have $\|v(.,s)\|_{L^2_\rho} = o(1/|s|)$ in all cases. Then using (2.42), we obtain $|\theta'(s)| \leq C/s^2$. Consequently, there exists a θ_0 such that $\theta(s) \to \theta_0$ as $s \to -\infty$. Using the definition (2.41), we get the convergence for w.
We will just prove (iii), since the proof for (i) and (ii) is the same and even easier.
Integrating the estimate for θ' (see (iii) of Proposition 2.3.5), we get

$$\theta(s) = \theta_0 - \frac{(1+\delta^2)\delta\kappa}{2(p-\delta^2)^2}\frac{1}{s} + O(\frac{\log|s|}{s^2}) \quad (2.51)$$

and

$$e^{i\theta(s)} = e^{i\theta_0}\left\{1 - i\frac{(1+\delta^2)\delta\kappa}{2(p-\delta^2)^2 s} + O(\frac{\log|s|}{s^2})\right\} \quad (2.52)$$

Using the fact that $w(y,s) = e^{i\theta(s)}(\kappa + v(y,s))$ (see(2.41)), the desired estimate follows from (2.52) and the L^2_ρ expansion of v from (ii) of Proposition 2.3.5. This concludes the proof of Proposition 2.2.5.■

Step 3 : Case where (i) of Proposition 2.2.5 holds
We prove Proposition 2.2.6, more precisely, we will prove that either $w \equiv \kappa e^{i\theta_0}$ or there exists $s_0 \in \mathbb{R}$, such that $w = \varphi_\delta(s - s_0)e^{i\theta_0}$.
As we wrote in Step 3 of section 2, if $\hat{\varphi}$ defined by (2.27), then we have

$$\forall s \leq \hat{s}, \ \ \|w(.,s) - \hat{\varphi}(s)e^{i\theta_0}\|_{L^2_\rho} \leq Ce^{3s/2}. \quad (2.53)$$

Our goal is to prove that $w \equiv \hat{\varphi}$ on $\mathbb{R} \times (-\infty, s_*]$. If we introduce $V = w - \hat{\varphi}e^{i\theta_0}$, then we see from (2.10) V satisfies :

$$\partial_s V = \left(\tilde{\mathcal{L}} + l(s)\right)V + B, \quad (2.54)$$

where

$$\tilde{\mathcal{L}}V = \Delta V - \frac{1}{2}y\cdot\nabla V + (1+i\delta)V, \ |l(s)| \leq Ce^s \text{ and } |B| \leq C|V|^2 \text{ for all } s \leq s_1. \quad (2.55)$$

As we saw in Lemma 2.19 and (2.22), $\tilde{\mathcal{L}}$ is diagonal with respect to $(\tilde{V}_1, \tilde{V}_2)$ such that $V = (1+i\delta)\tilde{V}_1 + i\tilde{V}_2$ and 1 is its largest eigenvalue.
Therefore, if we define $\|V\|_\smile = \sqrt{\int (\tilde{V}_1^2 + \tilde{V}_2^2)\rho}$, an equivalent norm to $\|V(.,s)\|_{L^2_\rho}$, then we get from (2.54) and (2.55)

$$\partial_s \|V\|_\smile \le (1 + Ce^s)\|V\|_\smile + C\|V^2\|_\smile.$$

To estimate $\|V^2\|_\smile$, it is easy to see from (2.54) and the fact that V is bounded that

$$\partial_s(|\tilde{V}_1| + |\tilde{V}_2|) \le \Delta(|\tilde{V}_1| + |\tilde{V}_2|) - \frac{y}{2}\cdot\nabla(|\tilde{V}_1| + |\tilde{V}_2|) + C(|\tilde{V}_1| + |\tilde{V}_2|).$$

Therefore, we can apply the regularizing effect of Lemma 2.3.4 to $|\tilde{V}_1| + |\tilde{V}_2|$ and obtain the existence of $C^* > 0$ and s^*, such that $\|V(.,s)^2\|_\smile \le C^*\|V(.,s-s^*)\|^2_\smile$. Then, we obtain

$$\forall s \le s_2,\ I'(s) \le \frac{5}{4}I(s) + CI(s-s^*)^2, \tag{2.56}$$

where $I(s) = \|V(.,s)\|_\smile$. Since $I(s) \le Ce^{3s/2}$ from (2.53), the following lemma from [MZ98a] allows us to conclude.

Lemma 2.3.6. *Consider $I(s)$ a positive $\mathcal{C}^1$ function such that (2.56) is satisfied and $0 \le I(s) \le Ce^{3s/2}$ for all $s \le s_2$, for some s_2. Then, for some $s_3 \le s_2$, we have $I(s) = 0$ for all $s \le s_3$.*

Proof : Trivial, left for the reader.■

Using Lemma 2.3.6, we see that $V \equiv 0$ on $\mathbb{R} \times (-\infty, s_3]$. Consequently, we have

$$\forall (y,s) \in \mathbb{R} \times (-\infty, s_3], \quad w(y,s) = \hat{\varphi}(s)e^{i\theta_0}. \tag{2.57}$$

From the uniqueness of the Cauchy problem for equation (2.10) and since w is defined for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, $\hat{\varphi}$ is also defined for $(y,s) \in \mathbb{R}\times\mathbb{R}$ and (2.57) holds for all $(y,s) \in \mathbb{R}\times\mathbb{R}$. Therefore, case (c) in (2.27) cannot hold and for all $(y,s) \in \mathbb{R}^2$, $w(y,s) = \kappa e^{i\theta_0}$ or $w(y,s) = \varphi_\delta(s-s_0)e^{i\theta_0}$. This concludes the proof of Proposition 2.2.6 and finishes Step 3.

Step 4 : Irrelevance of the case (iii) of Proposition 2.2.5
From the phase invariance of equation (2.10), we assume in Steps 4 and 5 that

$$\theta_0 = 0,$$

where θ_0 is given in Proposition 2.2.5.
As we said in step 4 of Section 2, it is enough to prove Proposition 2.2.8 (which we recall here) to conclude this case :
Proposition 2.2.8 *Assume that case (iii) from Proposition 2.2.5 holds, then there exists $\varepsilon_0 > 0$, such that :*

$$\begin{aligned} &\lim_{s\to-\infty} \sup_{|y|\le\varepsilon_0\sqrt{-s}} \left| w(y,s) - G\left(\tfrac{y}{\sqrt{-s}}\right)\right| = 0, \\ &\text{where } G(\xi) = \kappa\left(1 - \tfrac{(p-1)}{4(p-\delta^2)}\xi^2\right)^{-\frac{(1+i\delta)}{(p-1)}}. \end{aligned} \tag{2.58}$$

Indeed, let us first use Proposition 2.2.8 to find a contradiction ruling out case (iii) of Proposition 2.2.5, and then prove Proposition 2.2.8.
We define u_{s_0} by

$$u_{s_0}(\xi,\tau) = (1-\tau)^{-\frac{1+i\delta}{p-1}} w(y,s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}}{\sqrt{1-\tau}} \text{ and } s = s_0 - \log(1-\tau). \quad (2.59)$$

We note that u_{s_0} is defined for all $\tau \in [0,1)$ and $\xi \in \mathbb{R}$. u_{s_0} satisfies equation (2.2). The initial condition at time $\tau = 0$ is $u_{s_0}(\xi,0) = w(\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}, s_0)$. From (2.11), we have

$$\forall \tau \in [0,1),\ \|u_{s_0}(.,\tau)\|_{L^\infty} \le M(1-\tau)^{-\frac{1}{p-1}}. \quad (2.60)$$

Using Proposition 2.2.8, we get :

$$\sup_{|\xi|<4|s_0|^{1/4}} \left| u_{s_0}(\xi,0) - G(\frac{\varepsilon_0}{2}) \right| \equiv g(s_0) \to 0 \text{ as } s_0 \to -\infty.$$

If we define v, the solution of :

$$\begin{cases} v' &= (1+i\delta)|v|^{p-1}v, \\ v(0) &= G\left(\frac{\varepsilon_0}{2}\right), \end{cases}$$

then $v(\tau) = \kappa\left(1 - \frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)} - \tau\right)^{-\frac{(1+i\delta)}{p-1}}$, which blows up at time $1 - \frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)} < 1$. Therefore, there exists $\tau_0 < 1$, such that $|v(\tau_0)| = 2M(1-\tau_0)^{-\frac{1}{p-1}}$. Now, we consider the function $z = |\Re(u_{s_0} - v)| + |\Im(u_{s_0} - v)|$, then we have from Kato's inequality, for all $\tau \in [0,\tau_0]$ and $\xi \in \mathbb{R}$:

$$\partial_\tau z \le \Delta z + C(\varepsilon_0) z. \quad (2.61)$$

We recall Lemma 2.11 (page 1063) from [MZ98b] :

Lemma 2.3.7. *Assume that $z(\xi,\tau)$ satisfies for all $|\xi| \le 4B_1$ and $\tau \in [0,\tau_*]$:*

$$\begin{cases} \partial_\tau z &\le \Delta z + \lambda z + \mu, \\ z(\xi,0) &\le z_0,\ z(\xi,\tau) \le B_2, \end{cases}$$

where $\tau_* \le 1$. Then, for all $|\xi| \le B_1$ and $\tau \in [0,\tau_*]$,

$$z(\xi,\tau) \le e^{\lambda\tau}(z_0 + \mu + CB_2 e^{-B_1^2/4}).$$

Using the fact that z is bounded for all $\tau \in [0,\tau_0]$, by $B_2 = B_2(\varepsilon_0 = M(1-\tau_0)^{-\frac{1}{p-1}})$ (use (2.60)), we apply this Lemma with $B_1 = |s_0|^{1/4}$, $\tau_* = \tau_0$, $z_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$. Then, we get for all $\tau \in [0,\tau_0]$,

$$\sup_{|\xi| \le |s_0|^{1/4}} |z(\xi,\tau)| \le e^{C(\varepsilon_0)\tau_0}(g(s_0) + CB_2(\varepsilon_0)e^{-|s_0|^{1/2}/4}) \to 0 \text{ as } s_0 \to -\infty$$

(note that ε_0 and $\tau_0 = \tau_0(\varepsilon_0)$ are independent of s_0).
For $|s_0|$ large enough and $\xi = 0$, we get : $|z(0,\tau_0)| \leq \frac{M}{2}(1-\tau_0)^{-1/(p-1)}$ and

$$|u_{s_0}(0,\tau_0)| \geq |v(\tau_0)| - |z(0,\tau_0)| \geq \frac{3M}{2}(1-\tau_0)^{-\frac{1}{p-1}},$$

which is in contradiction with (2.60). Thus case (iii) of Proposition 2.2.5 cannot occur. Remains to prove Proposition 2.2.8.

Proof of Proposition 2.2.8 : We note $f(y,s) = G\left(\frac{y}{\sqrt{-s}}\right)$, then f satisfies

$$-\frac{y}{2}\cdot\nabla f - \frac{(1+i\delta)}{(p-1)}f + (1+i\delta)|f|^{p-1}f = 0.$$

Consider some arbitrary $\varepsilon_0 \in (0, R^*)$, where $R^* = \sqrt{\frac{4(p-\delta^2)}{(p-1)}}$. The parameter ε_0 will be fixed later in the proof small enough. If we note

$$F(y,s) = f(y,s) + (1+i\delta)\frac{\kappa}{2(p-\delta^2)}\frac{1}{s} - i\frac{(1+\delta^2)\delta\kappa^2}{(p-\delta^2)^2}\frac{1}{s}, \tag{2.62}$$

then, we see from (iii) of Proposition 2.2.5 that

$$\|\left(F(.,s) - w(.,s)\right)(1-\chi_{\varepsilon_0})\|_{L^2_\rho} = O\left(\frac{\log|s|}{s^2}\right) \text{ as } s \to -\infty, \tag{2.63}$$

where

$$\chi_{\varepsilon_0}(y,s) = 1 \text{ if } \frac{|y|}{\sqrt{|s|}} \geq 3\varepsilon_0 \text{ and zero otherwise.} \tag{2.64}$$

The formal idea of this proof is that F is an approximate solution (as $s \to -\infty$) of equation (2.10) satisfied by w. By (2.63), w and F are very close in the region $|y| \sim 1$. Our task is to prove that they remain close in the larger region $|y| \leq \varepsilon_0\sqrt{-s}$, for some ε_0 chosen later. Let us consider a cut-off function

$$\gamma(y,s) = \gamma_0\left(\frac{y}{\sqrt{-s}}\right), \tag{2.65}$$

where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\gamma_0(\xi) = 1$ if $|\xi| \leq 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \geq 4\varepsilon_0$. We introduce

$$\nu = (w - F). \tag{2.66}$$

We note $\nu = (1+i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$ and $Z = \gamma(|\tilde{\nu}_1| + |\tilde{\nu}_2|)$. Our proof is the same as Velázquez [Vel92], except for the fact that we need to perform a cut-off, since our profile $F(y,s)$ defined by (2.62) is singular on the parabola $y = R^*\sqrt{-s}$. The cut-off function will generate an extra term, difficult to handle. Let us present the major steps of the proof in the following. The proof of the presented Lemmas will be given at the end of this step.

Lemma 2.3.8. (Estimates in modified L^2_ρ spaces) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \le s_*$ and $y \in \mathbb{R}$:*

$$\partial_s Z - \Delta Z + \frac{1}{2} y \cdot \nabla Z - (1+\sigma) Z \le C\left(Z^2 + \frac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right) - 2\mathrm{div}\,((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)\,, \quad (2.67)$$

where $s_* \in \mathbb{R}$, $\sigma = 1/100$ and χ_{ε_0} is defined in (2.64). Moreover,

$$N^2_{2\varepsilon_0\sqrt{|s|}}(Z(s)) = o(1) \text{ as } s \to -\infty, \quad (2.68)$$

where the norm $N^q_r(\psi)$ is defined, for all $r > 0$ and $1 \le q < \infty$, by

$$N^q_r(\psi) = \sup_{|\xi| \le r} \left(\int |\psi(y)|^q \exp(-\frac{(y-\xi)^2}{4}) dy \right)^{1/q}. \quad (2.69)$$

Using the regularizing effect of the operator $\mathcal{L}$, we derive the following pointwise estimate, which allows us to conclude the proof of Proposition 2.2.8 :

Lemma 2.3.9. (An upper bound for $Z(y,s)$ in $\{|y| \le \varepsilon_0\sqrt{-s}\}$) *We have :*

$$\sup_{|y| \le \varepsilon_0\sqrt{-s}} Z(y,s) = o(1) \text{ as } s \to -\infty.$$

Indeed, we have by definition of Z, for all $|y| < \varepsilon_0\sqrt{-s}$, $|w - F| = \bar{\nu} \le CZ(y,s)$. Thus, Proposition 2.2.8 follows from Lemma 2.3.9. It remains to prove Lemma 2.3.8 and Lemma 2.3.9.

Proof of Lemma 2.3.8 : The proof of (2.67) is straightforward and a bit technical. We leave it to Appendix 2.5.2. Let us then prove (2.68). We take $s_0 < s_*$ and $s_0 \le s < s_*$ such that $e^{\frac{s-s_0}{2}} \le \sqrt{-s}$. We use the variation of constant formula in (2.67) to write

$$\begin{aligned} Z(y,s) \le\ & S_\sigma(s-s_0)Z(.,s_0) \\ & + \int_{s_0}^{s} S_\sigma(s-\tau)\left(C\left\{Z^2 + \frac{(y^2+1)}{\tau^2} + \chi_{\varepsilon_0}\right\} - 2\mathrm{div}\,((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)\right) d\tau, \end{aligned}$$

where S_σ is the semigroup associated to the operator $\mathcal{L}_\sigma \phi = \Delta\phi - \frac{1}{2} y \cdot \nabla\phi + (1+\sigma)\phi$, defined on $L^2_\rho(\mathbb{R})$. The kernel of the semigroup $S_\sigma(\tau)$ is

$$S_\sigma(\tau, y, z) = \frac{e^{(1+\sigma)\tau}}{(4\pi(1-e^{-\tau}))^{1/2}} \exp\left[-\frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})}\right]. \quad (2.70)$$

Setting

$$r \equiv r(s,s_0) = 2\varepsilon_0 e^{\frac{s-s_0}{2}} = R_1 e^{\frac{s-s_0}{2}} \quad (2.71)$$

and taking the N_r^2-norm we obtain

$$
\begin{aligned}
N_r^2(Z(.,s)) \le N_r^2(S_\sigma(s-s_0)Z(.,s_0)) + C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)Z(.,\tau)^2)d\tau \\
+C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)\left(\frac{(y^2+1)}{\tau^2}\right))d\tau \\
+C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)\chi_{\varepsilon_0}(y,\tau))d\tau + C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)(\mathrm{div}((|\tilde\nu_1|+|\tilde\nu_2|)\nabla\gamma))d\tau \\
\equiv J_1+J_2+J_3+J_4+J_5.
\end{aligned}
$$

In comparison with [Vel92], we have a new term J_5 coming from the cut-off terms. Therefore, we just recall in the following claim the estimates on $J_1 ... J_4$ from [Vel92], and treat J_5, which is a new ingredient in our proof :

Claim 2.3.10. *We obtain as $s \to -\infty$*

$$
\begin{aligned}
|J_1| &\le Ce^{(1+\sigma)(s-s_0)}\frac{\log|s_0|}{|s_0^2|},\\
|J_2| &\le C\int_{s_0}^{s_0+((s-R_0)-s_0)_+}\frac{e^{(1+\sigma)(s-\tau-R_0)}}{(1-e^{s-\tau-R_0})^{1/20}}\left(N_r^2(Z(.,\tau)^2)\right)d\tau + C\frac{e^{(s-s_0)(1+\sigma)}}{s_0^2},\\
&\quad \textit{with } R_0 = 4\varepsilon_0,\\
|J_3| &\le C\frac{e^{(s-s_0)(1+\sigma)}}{s_0^2}\left(1+(s-s_0)\right),\\
|J_4| &\le Ce^{(s-s_0)(1+\sigma)}e^{\alpha s}, \textit{ where } \alpha>0,\\
|J_5| &\le Ce^{(s-s_0)(1+\sigma)}e^{\beta s}, \textit{ where } \beta>0.
\end{aligned}
$$

Proof : See page 1578 in [Vel92] for $J_1 ... J_4$.

Now, we treat J_5. We have from (2.70) :

$$
\begin{aligned}
&S_\sigma(s-\tau)\left(-\mathrm{div}((|\tilde\nu_1|+|\tilde\nu_2|)\nabla\gamma)\right),\\
&= -\frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}}\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\mathrm{div}((|\tilde\nu_1|+|\tilde\nu_2|)\nabla\gamma)d\lambda,\\
&= \frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}}-\frac{(ye^{-(s-\tau)/2}-\lambda)}{2(1-e^{-(s-\tau)})}\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)(|\tilde\nu_1|+|\tilde\nu_2|)\nabla\gamma d\lambda.
\end{aligned}
\tag{2.72}
$$

Since w and F are bounded for $\frac{|y|}{\sqrt{-\tau}} \le R^*/2$ and $supp(\nabla\gamma) \subset (-4\varepsilon_0\sqrt{-\tau}, -3\varepsilon_0\sqrt{-\tau}) \cup (3\varepsilon_0\sqrt{-\tau}, 4\varepsilon_0\sqrt{-\tau})$, we have

$$
\begin{aligned}
|((|\tilde\nu_1|+|\tilde\nu_2|)\nabla\gamma\,| &\le C(|\tilde\nu_1|+|\tilde\nu_2|)\mathbb{I}_{\{3\varepsilon_0\le\frac{|y|}{\sqrt{-\tau}}\le 4\varepsilon_0\}},\\
&\le C(\mathbb{I}_{3\varepsilon_0\le\frac{|y|}{\sqrt{-\tau}}\le 4\varepsilon_0}) \le C\chi_{\varepsilon_0}.
\end{aligned}
$$

Using Cauchy-Schwartz inequality, we obtain :

$$|S_\sigma(s-\tau)\,(-\mathrm{div}((|\tilde{\nu}_1|+|\tilde{\nu}_2|)\nabla\gamma))| \leq \frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{3/2}}\mathcal{I}_1\mathcal{I}_2,$$

where,

$$\begin{aligned}
\mathcal{I}_1 &= \left(\int_{\mathbb{R}} (ye^{-(s-\tau)/2}-\lambda)^2 \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) d\lambda\right)^{1/2},\\
\mathcal{I}_2 &= \left(\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\chi_{\varepsilon_0} d\lambda\right)^{1/2}.
\end{aligned}$$

Doing a change of variables, we obtain $\mathcal{I}_1 = C(1-e^{-(s-\tau)})^{3/4}$. Furthermore, we have :

$$\mathcal{I}_2^2 \leq \mathcal{I}_3 \left(\int_{\mathbb{R}} \chi_{\varepsilon_0} e^{-\frac{\lambda^2}{4}} d\lambda\right)^{1/2},$$

where,

$$\mathcal{I}_3 = \left(\int \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{2(1-e^{-(s-\tau)})}+\frac{\lambda^2}{4}\right) d\lambda\right)^{1/2}.$$

We introduce $\theta = ye^{-(s-\tau)/2}$, by completing squares, we readily check that :

$$\frac{\lambda^2}{4} - \frac{(\theta-\lambda)^2}{2(1-e^{-(s-\tau)})} = -\frac{(1+e^{-(s-\tau)})}{4(1-e^{-(s-\tau)})}(\lambda - \frac{2\theta}{(1+e^{-(s-\tau)})})^2 + \frac{\theta^2}{2(1+e^{-(s-\tau)})},$$

then we obtain :

$$\mathcal{I}_3^2 = C\left(\frac{(1-e^{-(s-\tau)})}{(1+e^{-(s-\tau)})}\right)^{1/2} \exp\left(\frac{\theta^2}{2(1-e^{-(s-\tau)})}\right).$$

Therefore,

$$\begin{aligned}
\left|N_r^2(S_\sigma(s-\tau)\mathrm{div}((\tilde{\nu}_1+\tilde{\nu}_2)\nabla\gamma))\right| &\leq C\frac{e^{(s-\tau)(1+\sigma)}}{(1+e^{-(s-\tau)})^{1/8}(1-e^{-(s-\tau)})^{5/8}}\|\chi_{\varepsilon_0}\|^{1/2}\mathcal{I}_4,\\
\text{where } \mathcal{I}_4 &= N_r^2\left(\exp(\frac{y^2e^{-(s-\tau)}}{8((1-e^{-(s-\tau)}))})\right).
\end{aligned}$$

Let us compute $\mathcal{I}_4$. Using the fact that

$$\begin{aligned}
&-\frac{(y-\mu)^2}{4} + \frac{y^2e^{-(s-\tau)}}{4(1-e^{-(s-\tau)})} =\\
&\frac{1}{4}\left(-\left(y(1+e^{-(s-\tau)})^{-1/2} - \mu(1+e^{-(s-\tau)})^{1/2}\right)^2 + \mu^2e^{-(s-\tau)}\right),
\end{aligned}$$

and doing a change of variables, we obtain :

$$\begin{aligned}
&\int_{\mathbb{R}} \exp\left(-\frac{(y-\mu)^2}{4} + \frac{y^2e^{-(s-\tau)}}{4(1-e^{(s-\tau)})}\right) dy\\
&\leq C\exp\left(\frac{\mu^2e^{-(s-\tau)}}{4}\right)\int_{\mathbb{R}} \exp\left(-\frac{1}{4}\left(y(1+e^{-(s-\tau)})^{-1/2} - \mu(1+e^{-(s-\tau)})^{1/2}\right)^2\right) dy.
\end{aligned}$$

Hence $\mathcal{I}_4 \leq C(1+e^{-(s-\tau)})^{1/8}$ and

$$N_r^2(S_\sigma(s-\tau))\mathrm{div}((|\tilde{\nu}_1|+|\tilde{\nu}_2|)\nabla\gamma)) \leq C\frac{e^{(s-\tau)(1+\sigma)}}{(1-e^{-(s-\tau)})^{5/8}}\left(\int_{|\lambda|\geq R_1\sqrt{-\tau}} e^{-\frac{\lambda^2}{4}}d\lambda\right).$$

This gives

$$|J_5| = \int_{s_0}^{s} N_r^2\left(S_\sigma(s-\tau)(\mathrm{div}(|\tilde{\nu}_1|+|\tilde{\nu}_2|))\right)d\tau \leq C(\eta)e^{(s-s_0)(1+\sigma)}e^{\alpha s_0},$$

where $\alpha > 0$. This concludes the proof of the claim 2.3.10. ■
Summing up $J_{i=1..5}$, from claim 2.3.10 we obtain

$$N_r^2(Z(.,s)) \leq$$
$$e^{(s-s_0)(1+\sigma)}C\frac{\log|s_0|}{s_0^2} + C\int_{s_0}^{s_0+((s-R_0)-s_0)_+}\frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}}\left(N_r^2(Z(.,\tau))\right)^2 d\tau.$$

Now, we recall the following from [Vel92] :

Lemma 2.3.11. *Let ε, C, R, σ and α be positives constants, $0<\alpha<1$ and assume that $H(s)$ is a family of continuous functions satisfying :*

$$H(s) \leq \varepsilon e^{s(1+\sigma)} + C\int_0^{(s-R)_+}\frac{e^{(s-\tau)(1+\sigma)}H(\tau)^2}{(1-e^{(s-\tau-R)})^{\alpha}}d\tau \text{ for } s>0.$$

Then there exists $\xi = \xi(R,C,\alpha)$ such that for any $\varepsilon \in (0,\varepsilon_1)$ and any s for which $\varepsilon e^{s(1+\sigma)} \leq \xi$, we have

$$H(s) \leq 2\varepsilon e^{s(1+\sigma)}.$$

Proof : See the proof of Lemma 2.2 from [Vel92]. Note that the proof of [Vel92] is done in the case $\sigma = 0$, but it can be adapted to some $\sigma > 0$ with no difficulty.■

We conclude that $N^2_{r(\tau,s_0)}(Z(.,s)) \leq Ce^{(s-s_0)(1+\sigma)}\frac{\log|s_0|}{s_0^2}$ as $s \to -\infty$. If we fix $s = -e^{(s-s_0)}$, then we obtain $s \sim s_0$, $\log|s| \sim \log|s_0|$ and $N^2_{R_1\sqrt{-s}}(Z(.,s)) \leq Cs^{1+\sigma}\frac{\log|s_0|}{s_0^2} \leq C\frac{\log|s|}{s^{1-\sigma}} \to 0$ as $s \to -\infty$. Since $\sigma = \dfrac{1}{100}$, we get $N^2_{R_1\sqrt{-s}}(Z(.,s)) = o(1)$, as $s \to -\infty$.
This concludes the proof of Lemma 2.3.8.■

Proof of Lemma 2.3.9.
We aim at bounding $Z(y,s)$ for $|y| \leq R_2\sqrt{-s}$ in terms of $N_{R_1\sqrt{-s'}}(Z(s'))$, where $R_2 = \varepsilon_0$ and $R_1 = 2\varepsilon_0$, for some $s' < s$. Starting from equation (2.67), we do as in [Vel92] :

$$\begin{aligned} Z(.,s) \leq\ & \left\{e^{CR_0}S(R_0)Z(.,s-R_0)\right\} \\ & +\left\{C\int_{s-R_0}^{s} e^{C(s-\tau)}S(s-\tau)\left(\frac{(y^2+1)}{\tau^2}+\chi_{\varepsilon_0}\right)d\tau\right\} \\ & -\left\{2\int_{s-R_0}^{s} e^{C(s-\tau)}S(s-\tau)\left(\mathrm{div}((|\tilde{\nu}_1|+|\tilde{\nu}_2|)\nabla\gamma)\right)d\tau\right\} \\ =\ & \mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3, \text{ where } R_0 = 4\varepsilon_0, \end{aligned}$$

where S is the semigroup associated to the operator $\mathcal{L}$ defined in (2.35). The terms $\mathcal{M}_1$ and $\mathcal{M}_2$ are estimated in the following :

Claim 2.3.12. *(Velázquez) There exists s_0, such that for all $s \leq s_0$*

$$\begin{array}{rcl} \sup\limits_{|y|\leq R_2\sqrt{-s}} |\mathcal{M}_2| &=& \sup_{|y|\leq R_2\sqrt{-s}} \int_{s-R_0}^{s} \left(\frac{|y|^2+1}{s^2} + \chi_{\varepsilon_0}\right) \leq \frac{C}{|s|}, \\ \sup\limits_{|y|\leq R_2\sqrt{-s}} |\mathcal{M}_1| &=& \sup_{|y|\leq R_2\sqrt{-s}} |e^{CR_0} S(R_0) Z(., s-R_0)| = o(1) \ \textit{as}\ s \to -\infty. \end{array} \tag{2.73}$$

Proof : See page 1581 from [Vel92] and Lemma 6.5 in [HV93] in a similar case.■
It remains to estimate $\mathcal{M}_3$. Using page 57, we write

$$\begin{array}{l} |S(s-\tau)\left(-\mathrm{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)\right)| \\ = \left| \dfrac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}} \displaystyle\int_{\mathbb{R}} \exp\left(-\dfrac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) \mathrm{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma) d\lambda \right|, \\ = \left| \dfrac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}} \displaystyle\int_{\mathbb{R}} -\dfrac{(ye^{-(s-\tau)/2}-\lambda)}{2(1-e^{-(s-\tau)})} \exp\left(-\dfrac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) (|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma d\lambda \right|, \\ \leq \dfrac{Ce^{s-\tau}}{(1-e^{s-\tau})^{3/2}} \displaystyle\int_{\mathbb{R}} |ye^{-(s-\tau)/2}-\lambda| \exp\left(-\dfrac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda, \\ \leq \dfrac{Ce^{s-\tau}\sqrt{-\tau}}{(1-e^{s-\tau})^{3/2}} \displaystyle\int_{\mathbb{R}} \exp\left(-\dfrac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda. \end{array}$$

We make the change of variables $z = (1-e^{-(s-\tau)})^{-1/2}(\lambda - e^{-(\tau-s)/2}y)$ and we obtain

$$\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda \leq (1-e^{s-\tau})^{1/2} \int_{\Sigma} e^{-z^2/4} dz,$$

where,

$$\Sigma = \left\{ z \in \mathbb{R} : \left| z + e^{-(\tau-s)/2}(1-e^{s-\tau})^{-1/2} y \right| \geq 3\varepsilon_0 (1-e^{s-\tau})^{-1/2}\sqrt{-\tau} \right\}.$$

Since $|ye^{-(\tau-s)/2}| \leq \varepsilon_0\sqrt{-s}$, we readily see that $\Sigma \subset \left\{ z \in \mathbb{R} : |z| \geq \varepsilon_0\sqrt{-s} \right\}$. Then we conclude that

$$|S(s-\tau)\left(-\mathrm{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)\right)| \leq \frac{Ce^{s-\tau}}{(1-e^{s-\tau})} e^{\beta s}, \text{ where } \beta > 0,$$

and we obtain

$$\sup_{|y|\leq R_2\sqrt{-s}} |\mathcal{M}_3| = o(\frac{1}{|s|}) \text{ as } s \to -\infty.$$

Putting together $\mathcal{M}_{i=1..3}$, the proof of lemma 2.3.9 is complete. This concludes also the proof of Proposition 2.2.8 and rules out case (iii) of Proposition 2.2.5.■

Step 5 : Irrelevance of the case (ii) of Proposition 2.2.5
As we said in Step 4, we take

$$\theta_0 = 0,$$

where θ_0 is given in Proposition 2.2.5.
To conclude the proof of Theorem 1, we consider case (ii) of Proposition 2.2.5. We assume as in the previous case that $\theta_0 = 0$. We claim that the following proposition allows us to reach a contradiction in this case.

Proposition 2.3.13. *There exists $\varepsilon_0 > 0$ such that*

$$\lim_{s\to-\infty} \sup_{|y|\le\varepsilon_0 e^{-s/2}} \left|w(y,s) - G(ye^{s/2})\right| = 0, \text{ where } G(\xi) = \kappa(1 - C_1\kappa^{-p}\xi)^{-\frac{(1+i\delta)}{(p-1)}}. \quad (2.74)$$

Indeed, as in the previous Step, first, we will find a contradiction ruling out case (ii) of Proposition 2.2.5 and then prove Proposition 2.3.13.
We define u_{s_0} by

$$u_{s_0}(\xi,\tau) = (1-\tau)^{-\frac{1+i\delta}{p-1}} w(y,s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2}e^{-s_0/2}}{\sqrt{1-\tau}} \text{ and } s = s_0 - \log(1-\tau). \quad (2.75)$$

We note that u_{s_0} is defined for all $\tau \in [0,1)$ and $\xi \in \mathbb{R}$. u_{s_0} satisfies equation (2.2). The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2}e^{-s_0/2}, s_0)$. From (2.11), we have

$$\forall \tau \in [0,1),\ \|u_{s_0}(.,\tau)\|_{L^\infty} \le M(1-\tau)^{-\frac{1}{p-1}}. \quad (2.76)$$

Using Proposition 2.3.13, we get :

$$\sup_{|\xi|<4e^{-s_0/4}} |u_{s_0}(\xi,0) - G(\varepsilon_0/2)| \equiv g(s_0) \to 0 \text{ as } s_0 \to -\infty.$$

If we define v, the solution of :

$$\begin{cases} v' &= (1+i\delta)|v|^{p-1}v, \\ v(0) &= G(\frac{\varepsilon_0}{2}), \end{cases}$$

then $v(\tau) = \kappa\left(1 - C_1\kappa^{-p}\frac{\varepsilon_0}{2} - \tau\right)^{-\frac{(1+i\delta)}{p-1}}$, which blows up at time $1 - C_1\kappa^{-p}\frac{\varepsilon_0}{2} < 1$. Therefore, there exists $\tau_0 < 1$, such that $|v(\tau_0)| = 2M(1-\tau_0)^{-\frac{1}{p-1}}$. Now, we consider the function $z = |\Re(u_{s_0} - v)| + |\Im(u_{s_0} - v)|$, then we have from Kato's inequality for all $\tau \in [0,\tau_0]$ and $\xi \in \mathbb{R}$:

$$\partial_\tau z \le \Delta z + C(\varepsilon_0) z. \quad (2.77)$$

Using the fact that z is bounded for all $\tau \in [0,\tau_0]$ by $B_2 = B_2(\varepsilon_0) = M(1-\tau_0)^{-\frac{1}{p-1}}$ (use (2.76)), we use Lemma 2.3.7 with $B_1 = e^{-s_0/4}$, $\tau_* = \tau_0$, $z_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$. We obtain for all $\tau \in [0,\tau_0]$,

$$\sup_{|\xi|\le e^{-\varepsilon_0/4}} |z(\xi,\tau)| \le e^{C(\varepsilon_0)\tau_0}(g(s_0) + CB(\varepsilon_0)e^{-e^{-|s_0|/2}/4}) \to 0 \text{ as } s_0 \to -\infty$$

(note that ε_0 and $\tau_0 = \tau_0(\varepsilon_0)$ are independent of s_0).
For $|s_0|$ large enough and $\xi = 0$, we get $|z(0,\tau_0)| \le \frac{M}{2}(1-\tau_0)^{-\frac{1}{p-1}}$ and

$$|u_{s_0}(0,\tau_0)| \ge \frac{3}{2}M(1-\tau_0)^{-\frac{1}{p-1}},$$

which by (2.75) is in contradiction with (2.76) and case (ii) of Proposition 2.2.5 is ruled out. Now, we prove Proposition 2.3.13.

Proof of Proposition 2.3.13 : The proof is very similar to that of Proposition 2.2.8. We note $f(y,s) = G(ye^{s/2})$, then f satisfies

$$-\partial_s f - \frac{1}{2}y\cdot\nabla f - (1+i\delta)\frac{f}{(p-1)} + (1+i\delta)|f|^{p-1}f = 0. \tag{2.78}$$

Consider an arbitrary $\varepsilon_0 \in (0, \frac{R^*}{10})$, where $R^* = \dfrac{\kappa^p}{C_1}$. ε_0 will be fixed small enough later. Let us consider a cut-off function $\gamma(y,s) = \gamma_0(ye^{s/2})$, where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that $\gamma_0(\xi) = 1$ if $|\xi| \le 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \ge 4\varepsilon_0$. We note $\nu = (w-f)$ and $Z = \gamma\left(|\tilde\nu_1| + |\tilde\nu_2|\right)$. From (ii) of Proposition 2.2.5, we have

$$\|Z\| \le Ce^{s(1-\varepsilon)} \text{ as } s \to -\infty, \text{ for some } \varepsilon > 0. \tag{2.79}$$

As in the previous case, we Divide our proof in two parts given in the following lemmas.

Lemma 2.3.14. (Estimates in the modified L^2_ρ spaces.) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \le s_*$ and $y \in \mathbb{R}$,*

$$\partial_s Z - \Delta Z + \frac{1}{2}y\cdot\nabla Z - (1+\sigma)Z \le C(Z^2 + e^s + \chi_{\varepsilon_0}) - 2\text{div}((|\tilde\nu_1| + |\tilde\nu_2|)\nabla\gamma), \tag{2.80}$$

where $s_* \in \mathbb{R}$, $\sigma = \frac{1}{100}$ and

$$\chi_{\varepsilon_0}(y,s) = 1 \textit{ if } |y|e^{s/2} \ge 3\varepsilon_0 \textit{ and zero otherwise.} \tag{2.81}$$

Moreover, we have

$$N^2_{2\varepsilon_0 e^{-s/2}}(Z(s)) = o(1) \textit{ as } s \to -\infty. \tag{2.82}$$

As in Step 4, the following lemma allows us to conclude the proof of Proposition 2.3.13 :

Lemma 2.3.15. (An upper bound for $Z(y,s)$ in $|y| \le \varepsilon_0 e^{-s/2}$.) *We have :*

$$\sup_{|y|\le\varepsilon_0 e^{-s/2}} Z(y,s) = o(1) \textit{ as } s \to -\infty. \tag{2.83}$$

Remains to prove Lemmas 2.3.14 and 2.3.15 to conclude the proof of Proposition 2.3.13. Here, we only sketch the proof of Lemma 2.3.14, since it is completely similar to Step 4. We don't give the proof of Lemma 2.3.15. We refer the reader to Step 4 and Proposition 2.4 from Velázquez [Vel92] for similar situations.

Proof of Lemma 2.3.14 : As in the previous step, we leave the proof of (2.80) to Appendix 2.5.2.

Let us now apply variation of constants formula and take the norm $N^2_{r(s,s_0)}$, where $r(s,s_0)$ is as in (2.71). Assume that $s_0 < 2s_*$, then for all $s_0 \le s \le \frac{s_0}{2}$, we have

$$\begin{aligned} N^2_r(Z(.,s)) \;\le\;& N^2_r(S_\sigma(s-s_0)Z(.,s_0)) + C\textstyle\int_{s_0}^s N^2_r(S_\sigma(s-\tau)(Z(.,\tau)^2))d\tau \\ &+C\int_{s_0}^s N^2_r(S(s-\tau)(e^\tau))d\tau + C\int_{s_0}^s N^2_r(S_\sigma(s-\tau)(\chi_{\varepsilon_0}(.,\tau)))d\tau \\ &-2\int_{s_0}^s N^2_r(S_\sigma(s-\tau)(\text{div}((\tilde\nu_1| + |\tilde\nu_2|)\nabla\gamma)))d\tau \\ =\;& J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Arguing as in Step 4 and using (2.79), we prove :

Claim 2.3.16.

$$\begin{aligned}
|J_1| &\leq Ce^{(s-s_0)(1+\sigma)}e^{s_0(1-\varepsilon)},\\
|J_2| &\leq C\int_{s_0}^{s_0+((s-R_0)-s_0)_+}\frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}}\big(N_r^2(Z(.,s)^2)\big)d\tau + Ce^{(s-s_0)(1+\sigma)}e^{s}\\
\textit{with } R_0 &= 4\varepsilon_0,\\
|J_3| &\leq Ce^{(s-s_0)(1+\sigma)}e^{s},\\
|J_4| &\leq Ce^{(s-s_0)(1+\sigma)}e^{-\alpha e^{-s}} \textit{ where } \alpha>0,\\
|J_5| &\leq Ce^{(s-s_0)(1+\sigma)}e^{-\beta e^{-s}} \textit{ where } \beta>0.
\end{aligned}$$

Proof : To estimate $J_{i=1..4}$, see page 1584 in [Vel92]. To treat J_5, we proceed as in the proof of Lemma 2.3.8 of the previous Step.■
Summing up $J_{i=1..5}$, we obtain :

$$N_r^2(Z(.,s)) \leq Ce^{(s-s_0)(1+\sigma)}e^{(1-\varepsilon)s} + C\int_{s_0}^{s_0+((s-R_0)-s_0)_+}\frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}}\big(N_r^2(Z(.,s)^2)\big)d\tau,$$

then using Proposition 2.3.11, we get $N^2_{r(s,s_0)}(Z(.,s)) \leq Ce^{(s-s_0)(1+\sigma)}e^{(1-\varepsilon)s}$ as $s\to-\infty$ for $s_0\leq s\leq \frac{s_0}{2}$. If we fix $s=s_0/2$, then we obtain $N^2_{r(s,s_0)}(Z(.,s)) \leq Ce^{s(2(1-\varepsilon)-(1+\sigma))} \leq Ce^{s(1-(2\varepsilon+\sigma))} \to 0$ as $s\to-\infty$, since ϵ is small enough and $\sigma=\frac{1}{100}$. This concludes the proof of Lemma 2.3.14.■
As announced earlier, we don't give the proof of Lemma 2.3.15 and refer the reader to Step 4 and Section 2 from [Vel92]. This concludes the proof of Proposition 2.3.13 and rules out case (ii) of Proposition 2.2.5.

Conclusion of Part 3 and the sketch of proof of the Liouville theorem :
As we wrote in Section 2, we conclude from Step 4 and 5 that cases (ii) and (iii) of Proposition 2.2.5 are ruled out. By Step 3, we obtain that $w\equiv\kappa e^{i\theta_0}$ or $w\equiv\varphi_\delta(s-s_0)e^{i\theta_0}$ for some real s_0 and θ_0, where φ_δ is defined in Theorem 1, which is the desired conclusion of Theorem 1.

2.4 Applications of the Liouville Theorem for a type I blow-up solution of (2.2)

In this section we say how to adapt to the case $\delta\neq 0$, the proof of Proposition 3 given in [MZ98a] and [MZ00] in the case $\delta=0$.

Proof of (i) of Proposition 3 : The proof is exactly the same as in the case $\delta=0$ (see page 148 in [MZ98a]). However, one needs the following lower bound which is a bit tricky to get and which we give for the reader's convenience.

Lemma 2.4.1. *(Sharp lower bound on the blow-up rate) For all $t\in[0,T)$,*

$$\|u(t)\|_{L^\infty} \geq \kappa(T-t)^{-\frac{1}{p-1}}.$$

Remark : This bound is sharp, since there is equality for the solutions of the ODE $v' = (1+i\delta)|v|^{p-1}v$, which are particular solutions of (2.2).
Proof : We consider an arbitrary $\varepsilon > 0$ and introduce $\tilde\rho = \sqrt{\varepsilon + |u|^2}$, we claim that $\tilde\rho$ satisfies

$$\partial_t \tilde\rho \le \Delta\tilde\rho + \tilde\rho^p. \tag{2.84}$$

Indeed, we can easily prove that $\partial_t |u|^2 = \bar u \Delta u + u\Delta \bar u + 2|u|^{p+1}$. Then we have :

$$\begin{aligned}
\partial_t\tilde\rho &= \frac{\partial_t |u|^2}{2(\varepsilon + |u|^2)^{1/2}},\\
\Delta\tilde\rho &= \frac{\Delta |u|^2}{2(\varepsilon+|u|^2)^{1/2}} - \frac{|\nabla |u|^2|^2}{4(\varepsilon+|u|^2)^{3/2}},\\
&= \frac{\bar u\Delta u + u\Delta\bar u + 2|\nabla u|^2}{2(\varepsilon+|u|^2)^{1/2}} - \frac{\left|u\cdot\nabla\bar u + \bar u\cdot\nabla u\right|^2}{4(\varepsilon+|u|^2)^{3/2}}.
\end{aligned}$$

Using the fact that $\left|u\cdot\nabla\bar u + \bar u\cdot\nabla u\right|^2 \le 4|u|^2|\nabla u|^2 \le 4(\varepsilon+|u|^2)|\nabla u|^2$, we have $\Delta\tilde\rho \ge \dfrac{\bar u\Delta u + u\Delta\bar u}{2(\varepsilon+|u|^2)^{1/2}}$, hence

$$\partial_t\tilde\rho \le \Delta\tilde\rho + \frac{|u|^{p+1}}{(\varepsilon+|u|^2)^{1/2}} \le \Delta\tilde\rho + \tilde\rho^p,$$

which gives (2.84).
Now we prove that

$$\|\tilde\rho(t)\|_{L^\infty} \ge \kappa (T-t)^{-\frac{1}{p-1}}, \text{ for all } t\in[0,T). \tag{2.85}$$

For this, we argue by contradiction.
Assume that $\|\tilde\rho(t_0)\|_{L^\infty} < \kappa(T-t_0)^{-\frac{1}{p-1}}$, for some $t_0 < T$. Then, there exists $T_0 > T$ such that $\|\tilde\rho(t_0)\|_{L^\infty} \le \kappa(T_0-t_0)^{-\frac{1}{p-1}}$. Using the maximum principle, we get $\|\tilde\rho(t)\|_{L^\infty} \le \kappa(T_0-t)^{-\frac{1}{p-1}}$, for all $t\in[t_0,T)$, hence

$$\limsup_{t\to T} \|\tilde\rho(t)\|_{L^\infty} \le \kappa(T_0-T)^{-\frac{1}{p-1}} < \infty,$$

which is a contradiction. Therefore (2.85) holds. Making $\varepsilon\to 0$ in (2.85), we conclude the proof of Lemma 2.4.1.■

Proof of (ii) of Proposition 3 : Consider $|\delta| \le \delta_0$ and a solution $u(t)$ of (2.2), that blows up in finite time $T>0$ such that

$$\forall t\in[0,T),\ \|u(t)\|_{L^\infty} \le M(\delta)(T-t)^{-\frac{1}{p-1}}, \tag{2.86}$$

where δ_0 and $M(\delta)$ are defined in Theorem 1. Let us prove now the uniform pointwise control of the diffusion term by the nonlinear term, which asserts that the solution $u(t)$ behaves everywhere like the ODE $u' = (1+i\delta)|u|^{p-1}u$ (up to a constant).
The plan of the proof is the same as in [MZ98a] and [MZ00]. However, the Giga-Kohn

property "small local energy implies no blow-up locally" breaks down because we no longer have a gradient structure. The property has to be replaced by a new idea of ours "small L^2_ρ norm implies no blow-up locally" which is stated in Proposition 2.3.3.
We argue by contradiction and assume that for some $\varepsilon_0 > 0$, there exists $(x_n, t_n)_{n\in\mathbb{N}}$, a sequence of elements of $\mathbb{R}\times[\frac{T}{2}, T)$, such that

$$\forall n \in \mathbb{N},\ |\Delta u(x_n, t_n)| \geq \varepsilon_0 |u(x_n, t_n)|^p + n. \tag{2.87}$$

From the uniform estimates and the parabolic regularity, since $\|\Delta u\|_{L^\infty}$ is bounded on compact sets of $[\frac{T}{2}, T)$, we have

$$T - t_n \to 0, \text{ as } n \to \infty.$$

Part (i) of Proposition 3 implies that $|u(x_n, t_n)|(T-t_n)^{\frac{1}{p-1}}$ is uniformly bounded, therefore, we can assume that it converges as $n \to +\infty$. Let us consider two cases :

i)*Estimates in the very singular region :* $|u(x_n, t_n)|(T - t_n)^{\frac{1}{p-1}} \to \kappa_0 > 0$. From (2.87), it follows that

$$\|\Delta u(t_n)\|_{L^\infty} \geq |\Delta u(x_n, t_n)| \geq \varepsilon_0 \big(\frac{\kappa_0}{2}\big)^p (T - t_n)^{-\frac{p}{p-1}},$$

with $t_n \to T$, which contradicts (i) of Proposition 3.

ii)*Estimates in the singular region :* $u(x_n, t_n)(T - t_n)^{\frac{1}{p-1}} \to 0$.
We consider n large enough, such that

$$|u(x_n, t_n)|(T - t_n)^{\frac{1}{p-1}} \leq \frac{\eta_0}{3}, \text{ where } \eta_0 \text{ is defined in Proposition 2.3.3.}$$

We take $t_n^0 \to T$ such that

$$(T - t_n^0)^{-\frac{p}{p-1}} = \sqrt{n}. \tag{2.88}$$

Using (2.87) and uniform estimates, we obtain :

$$n \leq |\Delta u(x_n, t_n)| \leq C_0 (T - t_n)^{-\frac{p}{p-1}},$$

hence $t_n^0 < t_n$. Now we distinguish two cases :
Case 1. We assume that (up to extracting a subsequence) there exists $t_n' \in (t_n^0, t_n)$, such that $|u(x_n, t_n')|(T - t_n')^{\frac{1}{p-1}} = \frac{2}{3}\eta_0$. If we consider

$$v_n(\xi, \tau) = (T - t_n')^{\frac{1}{p-1}} u(x_n + \xi\sqrt{T - t_n'}, t_n' + \tau(T - t_n')), \tag{2.89}$$

then, we have from (i) of Proposition 3 and (2.86)

$$|v_n(0,0)| = \frac{2}{3}\eta_0,\ \|\nabla v_n(0)\|_{L^\infty} + \|\Delta v_n(0)\|_{L^\infty} \to 0, \tag{2.90}$$

$$\forall \tau < 1,\ \|v_n(\tau)\|_{L^\infty} \leq M(\delta)(1-\tau)^{-\frac{1}{p-1}} \text{ and } \partial_\tau v_n = \Delta v_n + (1 + i\delta)|v_n|^{p-1} v_n.$$

Using parabolic regularity, we can extract a subsequence (still denoted by t_n) such that, $v_n(\xi,\tau) \to \hat{v}(\xi,\tau)$ in $\mathcal{C}^{2,1}$ of every compact set of $\mathbb{R}\times(-\infty,1)$, with

$$\partial_\tau \hat{v} = \Delta \hat{v} + (1+i\delta)|\hat{v}|^{p-1}\hat{v},\ |\hat{v}(0,0)| = 2/3\eta_0 \text{ and } \|\hat{v}\|_{L^\infty} \le M(\delta)(1-\tau)^{-\frac{1}{p-1}}.$$

Using the Liouville Theorem (see Theorem 2), we get

$$\hat{v}(\xi,\tau) = \kappa\left(\left(\frac{3\kappa}{2\eta_0}\right)^{p-1} - \tau\right)^{-\frac{1+i\delta}{p-1}} e^{i\theta_0}, \text{ for some } \theta_0 \in \mathbb{R}.$$

We claim that it is enough to extend the convergence of $v_n \to \hat{v}$ to all $\tau \in [0,1)$ (and $\xi = 0$), to conclude. Indeed, if we have this extended convergence, then we write from (2.87) and the definition (2.89) of v_n,

$$|\Delta v_n(0,\tau_n)| = (T-t'_n)^{\frac{p}{p-1}}|\Delta u(0,t_n)| \ge \frac{\varepsilon_0}{2}|u(0,t_n)|^p(T-t'_n)^{\frac{p}{p-1}} \ge \frac{\varepsilon_0}{2}|v_n(0,\tau_n)|^p,$$

with $\tau_n = \frac{t_n - t'_n}{T-t'_n}$. Letting $n\to\infty$, we obtain

$$0 \ge \frac{\varepsilon_0}{2}\min_{\tau\in[0,1]}|\hat{v}(\tau)|^p \ge \frac{\varepsilon_0}{2}\left(\frac{2}{3}\eta_0\right)^p, \tag{2.91}$$

which is a contradiction.
Let us then extend the convergence. If we consider the following similarity variables,

$$y = \frac{\xi-\xi_0}{\sqrt{1-\tau}},\ s = -\log(1-\tau),\ w_{n,\xi_0}(y,s) = (1-\tau)^{\frac{1}{p-1}}v_n(\xi,\tau), \tag{2.92}$$

then, we see from (2.90) that for all $|\xi_0|\le 1$, $\|w_{n,\xi_0}(.,0)\|_{L^2_\rho} \le \eta_0$, for n large enough. Using Proposition 2.3.3, we get for all $|\xi|\le 1$ and $\tau\in[0,1)$, $|v_n(\xi,\tau)| \le M_0$. Using the parabolic regularity, we can extend the convergence, and then reach the contradiction (2.91). This concludes Case 1.

Case 2. We assume that for some $n_0\in\mathbb{N}$, for all $n\ge n_0$ and $t\in[t_n^0, t_n]$, we have :

$$(T-t)^{\frac{1}{p-1}}|u(x_n,t)| < \frac{2}{3}\eta_0.$$

Then, we take $t'_n = t_n^0$ and introduce v_n by (2.89). As in Case 1, we obtain by Proposition 2.3.3 and the parabolic regularity :

$$\forall |\xi|\le 1 \text{ and } \tau\in[0,1),\ |v_n(\xi,\tau)|\le M_0,\ \ |\Delta v_n(0,\tau_n)| \le C_0\eta_0 \text{ where } \tau_n = \frac{t_n - t_n^0}{T-t_n^0}.$$

Therefore, we get from (2.87), (2.89) and (2.88) :

$$n \le |\Delta u_n(x_n,t_n)| = (T-t_n^0)^{-\frac{p}{p-1}}|\Delta v_n(0,\tau_n)| \le C_0\eta_0(T-t_n^0)^{-\frac{p}{p-1}} = C_0\eta_0\sqrt{n},$$

which is a contradiction, as $n\to\infty$. This ends Case 2 and concludes the proof of Proposition 3.■

2.5 Appendix

2.5.1 Proof of Proposition 2.3.5

We prove Proposition 2.3.5 here, we recall from (2.44,..,2.47) :

$$\tilde{v}_{1s} = \mathcal{L}\tilde{v}_1 + \theta'(s)(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1, \tag{2.93}$$
$$\tilde{v}_{2s} = (\mathcal{L}-1)\tilde{v}_2 - \theta'(s)((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2, \tag{2.94}$$

where $\mathcal{L}$ is given in (2.35) and

$$\tilde{G}_1 = \frac{p-\delta^2}{2\kappa}\tilde{v}_1^2 + \frac{1}{2\kappa}\tilde{v}_2^2 + O(|v|^3), \tag{2.95}$$
$$\tilde{G}_2 = (1+\delta^2)\frac{\tilde{v}_1(\delta\tilde{v}_1 + \tilde{v}_2)}{\kappa} + O(|v|^3). \tag{2.96}$$

A primary idea to deal with system (2.93,..,2.96) is to confirm that it is driven by its linear part $\partial_s(\tilde{v}_1, \tilde{v}_2) = (\mathcal{L}\tilde{v}_1, (\mathcal{L}-1)\tilde{v}_2)$ (except for the neutral modes $\tilde{v}_{12}$ where the second order terms matter, and $\tilde{v}_{20} = 0$ by the choice of the modulation parameter ; see (2.24)).
To this end, let us decompose $\tilde{v}_1$ and $\tilde{v}_2$, respectively with respect to the spectrum of $\mathcal{L}$ (with a positive ($\lambda = 1$ or $\lambda = 1/2$), zero and nonnegative part ($\lambda \leq -1/2$)) and $\mathcal{L}-1$ (with zero eigenvalue and a nonnegative part ($\lambda \leq -1/2$)). Let us introduce some notations

$$\tilde{v}_{1+}(y,s) = \tilde{v}_{10}(s)h_0(y) + \tilde{v}_{11}(s)h_{11}(y),\ z(s) = \|\tilde{v}_{1+}(.,s)\|_{L^2_\rho},$$
$$\tilde{v}_{1null}(y,s) = \tilde{v}_{12}(s)h_2(y),\ x(s) = \|\tilde{v}_{1null}(.,s)\|_{L^2_\rho},$$
$$\tilde{v}_{1-}(y,s) = \textstyle\sum_3^{+\infty} \tilde{v}_{1m}(s)h_m(y),\ y_1(s) = \|\tilde{v}_{1-}(.,s)\|_{L^2_\rho},$$

and we note by

$$\tilde{v}_{2\perp}(y,s) = \textstyle\sum_1^{+\infty} \tilde{v}_{2m}(s)h_m(y), \quad y_2(s) = \|\tilde{v}_{2\perp}(.,s)\|_{L^2_\rho}.$$

Since we have $\tilde{v}_{20}(s) = 0$ from (2.24), it follows that

$$\tilde{v}_{2\perp}(y,s) = \tilde{v}_2(y,s) \text{ and } y_2(s) = \|\tilde{v}_2(.,s)\|_{L^2_\rho}.$$

Finally, we note

$$N_1(s) = \|\theta_s(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1\|_{L^2_\rho},$$
$$N_2(s) = \|\theta_s((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2\|_{L^2_\rho},$$

We proceed in 3 steps :
- In step 1, we use ODE techniques to show that either z or x dominates as $s \to -\infty$.
- In step 2, we consider the case where z dominates and show that it leads to case (i) or (ii) of Proposition 2.3.5.
- In step 3, we show that (iii) of Proposition 2.3.5 holds in the case where x dominates.

Step 1 : Either $\|\tilde{v}_{1+}(.,s)\|_{L^2_\rho}$ or $\|\tilde{v}_{1null}(.,s)\|_{L^2_\rho}$ dominates as $s \to -\infty$

Projecting (2.93) onto the unstable subspace of $\mathcal{L}$ forming the L^2_ρ-inner product with $\tilde{v}_{1+}$, and using standard inequalities, we get

$$\dot{z} \geq \frac{1}{2}z - N_1.$$

Working similarly with $\tilde{v}_{10}(s)$, $\tilde{v}_{1-}(y,s)$ and $\tilde{v}_2(y,s)$ we arrive at the system

$$\begin{array}{lll} \dot{z} & \geq & \frac{1}{2}z - N_1, \\ |\dot{x}| & \leq & N_1, \\ \dot{y_1} & \leq & -\frac{1}{2}y_1 + N_1, \\ \dot{y_2} & \leq & -\frac{1}{2}y_2 + N_2. \end{array} \tag{2.97}$$

Using the fact that v is bounded (see (2.11) and (2.14)), and (2.15), we obtain easily

$$N_1^2 + N_2^2 \leq C \int |v|^4 \rho, \tag{2.98}$$

for some positive constant C. Thus, it follows from (2.97) that

$$\begin{array}{lll} \dot{z} & \geq & \frac{1}{2}z - CN \\ |\dot{x}| & \leq & CN \\ \dot{y} & \leq & -\frac{1}{2}y + CN, \end{array} \tag{2.99}$$

where

$$y \equiv y_1 + y_2 \text{ and } N^2 \equiv \int |v|^4 \rho. \tag{2.100}$$

If we knew that for $|s|$ large enough

$$N \leq \varepsilon(x + y + z), \tag{2.101}$$

which is equivalent to $\int |v|^4 \rho \leq \varepsilon^2 \int |v|^2 \rho$, we could use ODE techniques to conclude the step. The meaning of estimate (2.101) is essentially that the $L^2_\rho-$norm of quadratic term $|v|^2$ is small compared to the norm of the linear term $|v|$. However, we do not have this information at this stage. We thus estimate N as follows. Given any $\varepsilon > 0$, and any $\alpha > 0$ (both will be chosen small in the sequel), there is a time s_* such that :

$$\int |v|^4 \rho = \int_{|y|>\alpha^{-1}} |v|^4 \rho + \int_{|y|<\alpha^{-1}} |v|^4 \rho \leq \alpha^k \int |v|^4 |y|^k \rho + \varepsilon^2 \int |v|^2 \rho \text{ for all } s \leq s_*. \tag{2.102}$$

Here we use the fact that $v(y,s)$ goes to zero uniformly on the compact set $|y| < \alpha^{-1}$, which follows from (ii) of Lemma 2.2.3 and parabolic regularity. The exponent k which appears in (2.102) is an arbitrary positive integer (later we will choose it to be large). We set

$$J^2 \equiv \int |v|^4 |y|^k \rho,$$

so that (2.102) can be rewritten as

$$\int |v|^4\rho \le \alpha^k J^2 + \varepsilon^2 \int |v|^2\rho \text{ for all } s \le s_*.$$

From the inequalities above, we get that

$$N \le \alpha^{k/2} J + \varepsilon(x+y+z) \text{ for all } s \le s_*. \tag{2.103}$$

We next estimate J. Multiplying (2.93) by $\tilde{v}_1|v|^2|y|^k\rho$, and (2.45) by $\tilde{v}_2|v|^2|y|^k\rho$, integrating over all $\mathbb{R}$, we get after some calculations :

$$\dot{J} \le -\theta J + \varepsilon^{'}(x+y+z) + c(x+y+z)^2,$$

where

$$\theta = \frac{k}{4} - c - \frac{k\alpha^2}{2}(k-1) \text{ and } \varepsilon' = \frac{1}{2}\varepsilon\alpha^{2-k/2}k(k+n-2). \tag{2.104}$$

Using the fact that $x, y, z \to 0$ as $s \to -\infty$, we end up with

$$\dot{J} \le -\theta J + 4\varepsilon'(x+y+z), \tag{2.105}$$

where θ is still given by (2.104) with a different value of the constant c, to end the proof we choose k large enough (certainly $k > 4$), so that for some $\alpha^*(k) > 0$, we have for $0 < \alpha < \alpha^*$, $\theta \ge \frac{1}{2}$. We obtain from (2.99), (2.103) and (2.105) :

$$\begin{array}{rcl} \dot{z} & \ge & (\frac{1}{2} - \hat{\varepsilon}) - \hat{\varepsilon}(x+\tilde{y}), \\ |\dot{x}| & \le & \hat{\varepsilon}(x+\tilde{y}+z), \\ \dot{\tilde{y}} & \le & -(\frac{1}{2} - \hat{\varepsilon})\tilde{y} + \hat{\varepsilon}(x+z) \end{array}$$

where

$$\tilde{y} \equiv y + J,\ \hat{\epsilon} \equiv C\max(\varepsilon + \varepsilon\alpha^{2-k/2}, \alpha^{k/2}).$$

Note that $\hat{\varepsilon}$ can be made arbitrarily small by choosing first α and then ε sufficiently small. Now, we conclude using the following lemma :

Lemma 2.5.1. *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions that are non negative and satisfy :*
i) $(x, y, z)(s) \to 0$ as $s \to -\infty$,
ii) For some $c_0 \in \mathbb{R}$ and for all $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for all $s \le s_0$

$$\begin{array}{rcl} \dot{z} & \ge & c_0 z - \varepsilon(x+y) \\ |\dot{x}| & \le & \varepsilon(x+y+z) \\ \dot{y} & \le & -c_0 y + \varepsilon(x+z). \end{array} \tag{2.106}$$

Then either $x + y = o(z)$ or $y + z = o(x)$, as $s \to -\infty$.

Proof : Here, we adapt the proof of Lemma A.1 (page 172) from [MZ98a]. By rescaling in time, we may assume $c_0 = 1$.
Part 1. Let $\varepsilon > 0$. We show in this part that either

$$\exists s_2(\varepsilon) \text{ such that } \forall s \le s_2,\ z(s) + y(s) \le C\varepsilon x(s), \tag{2.107}$$

or

$$\exists s_2(\varepsilon) \text{ such that } \forall s \leq s_2,\ x(s) + y(s) \leq C\varepsilon z(s). \tag{2.108}$$

We show that for all $s \leq s_0(\varepsilon)$, $\beta(s) \leq 0$ where $\beta = y - 2\varepsilon(x+z)$. We argue by contradiction and suppose that there exists $s_* \leq s_0(\varepsilon)$ such that $\beta(s_*) > 0$. Then, if $s \leq s_*$ and $\beta(s) > 0$, we have from (2.106) $\dot\beta = \dot y - 2\varepsilon(\dot x + \dot z) \leq 0$. Therefore, for all $s \leq s_*$, $\beta(s) \geq \beta(s_*) > 0$, which contradicts $\beta(s) \to 0$ as $s \to -\infty$. Thus, for all $s \leq s_0(\varepsilon)$

$$y \leq 2\varepsilon(x+z). \tag{2.109}$$

Therefore, (2.106) yields

$$\dot z \geq \frac{1}{2}z - 2\varepsilon x, \tag{2.110}$$
$$|\dot x| \leq 2\varepsilon(x+z). \tag{2.111}$$

Let $\gamma(s) = 8\varepsilon x(s) - z(s)$. Two cases then arise :

- Case 1. There exists $s_2 \leq s_0(\varepsilon)$ such that $\gamma(s_2) > 0$. Then we compute $\dot\gamma = 8\varepsilon\dot x - \dot z \leq 16\varepsilon^2(x+z) - \frac{1}{2}z + 2\varepsilon x = \gamma(s)\left(\frac{1}{4} + 2\varepsilon\right) - z(s)\left(\frac{1}{4} - 2\varepsilon - 16\varepsilon^2\right)$. Therefore, for all $s \leq s_2$, $\gamma(s) \geq \gamma(s_2)e^{\left(\frac{1}{4}-2\varepsilon\right)(s-s_2)} > 0$, that is, $8\varepsilon x(s) > z(s)$. Together with (2.109), this yields (2.107).
- Case 2. For all $s \leq s_0(\varepsilon)$, $\gamma(s) \leq 0$, that is, $8\varepsilon x(s) \leq z(s)$. In this case (2.111) yields

$$\forall s \leq s_0(\varepsilon),\ \dot z \geq \frac{1}{4}z \text{ and } \dot x \leq \left(2\varepsilon + \frac{1}{4}\right)z, \text{ hence } \dot x \leq (1+8\varepsilon)\dot z. \tag{2.112}$$

  By integration, we get $x(s) \leq (8\varepsilon+1)z(s)$. We inject this in (2.111) and get from (2.112) $\dot x \leq 2\varepsilon(x+z) \leq 2\varepsilon z(2+8\varepsilon) \leq 8\varepsilon(2+8\varepsilon)\dot z(s)$ which gives $x(s) \leq 8\varepsilon(2+8\varepsilon)z(s)$ by integration.

Part 2. It is easy to see that if for some $\varepsilon > 0$, (2.107) holds, then it holds for all $\varepsilon' < \varepsilon$ and the same with (2.108). This concludes the proof of Lemma 2.5.1.■

Applying Lemma 2.5.1, we get either

$$\|\tilde v_{12}\|_{L^2_\rho} + \|\tilde v_{1-}(.,s)\|_{L^2_\rho} + \|\tilde v_2(.,s)\|_{L^2_\rho} = o(\|\tilde v_{1+}(.,s)\|_{L^2_\rho})$$

or

$$\|\tilde v_{1+}(.,s)\|_{L^2_\rho} + \|\tilde v_{1-}(.,s)\|_{L^2_\rho} + \|\tilde v_2(.,s)\|_{L^2_\rho} = o(\|\tilde v_{12}\|_{L^2_\rho}).$$

Step 2 : Case where $\|\tilde v_{1+}(.,s)\|_{L^2_\rho}$ dominates

Now, we focus on the case $\|\tilde v_{1null}(.,s)\|_{L^2_\rho} + \|\tilde v_{1-}(.,s)\|_{L^2_\rho} + \|\tilde v_2(.,s)\|_{L^2_\rho} = o(\|\tilde v_{1+}(.,s)\|_{L^2_\rho})$. We will show that it leads to either case (i) or case (ii) of Proposition 2.3.5. We want to derive from (2.93) the equations satisfied by $\tilde v_{10}$ and $\tilde v_{11}$. For this, we estimate in the following lemma, $\int \tilde G_1 k_m(y)\rho(y)dy$ for $m = 0, 1$ where

$$k_m(y) = h_m(y)/\|h_m\|^2_{L^2_\rho}$$

and $\tilde G_1$ is given by (2.95).

Lemma 2.5.2. *There exists $\beta_0 > 0$, and an integer $k' > 4$ such that for all $\beta \in (0, \beta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int v^2|y|^{k'}\rho \leq c_0(k')\beta^{4-k'}z(s)^2$.*

Proof : This lemma is analogous to Lemma A.3 p 175 from [MZ98a], which handles the real case with $\delta = 0$. One can adapt with no difficulty the proof of the present context. ■

Proceeding as in Appendix.A from [MZ98a] and doing the projection of equation (2.44), respectively on $k_0(y)$ and $k_1(y)$, we obtain

$$\tilde{v}'_{10}(s) = \tilde{v}_{10}(s) + \frac{p-\delta^2}{2\kappa}(1+\alpha(s))z^2(s), \tag{2.113}$$

and

$$\tilde{v}'_{11}(s) = \frac{1}{2}\tilde{v}_{11}(s) + \eta(s)z(s)^2, \tag{2.114}$$

where $z(s) = \|\tilde{v}_{1+}(.,s)\|_{L^2_\rho}$, $\alpha(s) \to 0$ as $s \to +\infty$ and η is bounded. Then, from standard ODE techniques, we get

$$\forall \varepsilon > 0,\ \tilde{v}_{10}(s) = O(e^{(1-\varepsilon)s}) \text{ and } \tilde{v}_{11} = C_1 e^{\frac{s}{2}} + O(e^{(1-\varepsilon)s}). \tag{2.115}$$

Since $z(s)^2 = \|\tilde{v}_{1+}(.,s)\|^2_{L^2_\rho} = \tilde{v}_{10}^2 + 2\tilde{v}_{11}^2$, we write (2.113) as

$$\tilde{v}'_{10}(s) = \tilde{v}_{10}(s) + \frac{p-\delta^2}{\kappa}|C_1|^2 e^s(1+\alpha(s)) + \gamma(s), \tag{2.116}$$

where $\gamma(s) = O(e^{(\frac{3}{2}-\varepsilon)s})$ and $\alpha(s) \to 0$ as $s \to -\infty$, which gives by integration

$$\tilde{v}_{10}(s) = \frac{p-\delta^2}{\kappa}|C_1|^2 s e^s(1+o(s)) + C_0 e^s + O(e^{(\frac{3}{2}-\varepsilon)s}), \text{ as } s \to -\infty. \tag{2.117}$$

Two cases then arise :

- If $C_1 \neq 0$, then $\tilde{v}_{11} \equiv C_1 e^{\frac{s}{2}} \gg \tilde{v}_{10} = O(se^s)$, from (2.117). Note first that applying Lemma 2.3.4 to $|v_1| + |v_2|$ (this is possible from equations (2.93) and (2.94) and the boundedness of v), we have for all $|s|$ large enough (and $s < 0$),

$$N^2 = \int |v(y,s)|^4\rho(y)dy \leq C * \|v(.,s-s^*)\|^2_{L^2_\rho}, \tag{2.118}$$

  for some positive s^* and C^*.
  Recalling system (2.99) and using (2.118), we obtain, $\dot{y} \leq -\frac{1}{2}y + c\|v(.,s-s^*)\|^2_{L^2_\rho} \leq -\frac{1}{2}y + ce^s$. Then, we obtain $y = O(e^s)$, similarly, we obtain $x = \|\tilde{v}_{1null}(.,s)\|_{L^2_\rho} = O(e^s)$. We conclude that $\|v(.,s) - (1+i\delta)C_1 e^{s/2}y\|_{L^2_\rho} = O(e^{s(1-\varepsilon)})$ as $s \to -\infty$, for some $\varepsilon > 0$. Using (2.15), we get $|\theta_s| \leq Ce^s$. This is case (ii) of Proposition 2.3.5.
- If $C_1 = 0$, we obtain case (i) of Proposition 2.3.5. Indeed, let us first improve the estimate of v. In fact in (2.116) we have $\gamma(s) = O(e^{2(1-\varepsilon)s})$. Hence, arguing as for (2.117), we get $\tilde{v}_{10} = C_0 e^s + O(e^{3s/2})$ and from (2.114) $\tilde{v}_{11} = O(e^{3s/2})$.

We note $y = \|\tilde{v}_{1-}(.,s)\|_{L^2_\rho} + \|\tilde{v}_2(.,s)\|_{L^2_\rho}$ and $x = \|\tilde{v}_{1null}(.,s)\|_{L^2_\rho}$. Recalling system (2.99) and using (2.118), we obtain

$$\dot{y} \leq -\frac{1}{2}y + c\|v(.,s-s^*)\|^2_{L^2_\rho} \leq -\frac{1}{2}y + ce^{2s}.$$

Then, we have that $y = \|\tilde{v}_{1-}(.,s)\|_{L^2_\rho} + \|\tilde{v}_2(.,s)\|_{L^2_\rho} = O(e^{3s/2})$. Similarly, we obtain that

$$x = \|\tilde{v}_{1null}(.,s)\|_{L^2_\rho} = O(e^{3s/2})$$

and we conclude

$$\begin{aligned}&\|v(.,s) - (1+i\delta)\tilde{v}_{10}(s)\|_{L^2_\rho}\\ &= \|(1+i\delta)\left(\tilde{v}_{11}(s) + \tilde{v}_{1,null}(.,s) + \tilde{v}_{1-}(.,s)\right) + i\tilde{v}_2(.,s)\|_{L^2_\rho}\\ &= O(e^{3s/2}).\end{aligned}$$

Using (2.15), we get $|\theta'(s)|_{L^2_\rho} \leq Ce^{2s}$. This is case (i) of Proposition 2.3.5.

Step 3 : Case where $\|\tilde{v}_{1null}(.,s)\|_{L^2_\rho}$ dominates

In the following we prove that (iii) of Proposition 2.3.5 holds. First, we prove the following Lemma :

Lemma 2.5.3. *Assume that*

$$\|\tilde{v}_{1+}(.,s)\|_{L^2_\rho} + \|\tilde{v}_{1-}(.,s)\|_{L^2_\rho} + \|\tilde{v}_2(.,s)\|_{L^2_\rho} = o(\|\tilde{v}_{1null}(.,s)\|_{L^2_\rho}) \qquad (2.119)$$

holds. Then

$$v(y,s) = -(1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2) + o(\frac{1}{s}),$$

in L^2_ρ as $s \to -\infty$.

Proof : Since $\tilde{v}_{1null} = \tilde{v}_{12}(s)h_2(y)$, we note that $\tilde{v}_{12} = \int \tilde{v}_1 k_2 \rho$. Projecting equation (2.93) onto $h_2(y)$ we get

$$\begin{aligned}\frac{d}{ds}(\tilde{v}_{12}) &= \frac{p-\delta^2}{2\kappa}\int \tilde{v}_1^2 k_2(y)\rho(y)\\ &\quad +\theta'(s)\int(\delta\tilde{v}_1 + \tilde{v}_2)k_2(y)\rho(y) + \int \frac{1}{2\kappa}\tilde{v}_2^2 k_2(y)\rho(y) + O(\int |v|^3 k_2(y)\rho(y)),\\ &= \frac{p-\delta^2}{2\kappa}\int \tilde{v}_{1null}^2 k_2(y)\rho(y) - \frac{p-\delta^2}{2\kappa}\int(\tilde{v}_{1null}^2 - \tilde{v}_1^2)k_2(y)\rho(y)\\ &\quad +\theta'(s)\int(\delta\tilde{v}_1 + \tilde{v}_2)k_2(y)\rho(y) + \int \frac{1}{2\kappa}\tilde{v}_2^2 k_2(y)\rho(y) + O(\int |v|^3 k_2(y)\rho(y)),\\ &\equiv \frac{(p-\delta^2)}{2\kappa}8\tilde{v}_{12}^2 + \frac{p-\delta^2}{2\kappa}\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4,\end{aligned}$$

where we use the fact that $\int \tilde{v}_{null} k_2 \rho = \tilde{v}_{12}^2 \int h_2^2 k_2 \rho = 8\tilde{v}_{12}^2$. We next estimate $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ and $\mathcal{E}_4$. For this, we need the following lemma :

Lemma 2.5.4. *There exist $\alpha_0 > 0$ and an integer $k' > 4$ such that for all $\alpha \in (0, \alpha_0)$, there exists $s_0 \in \mathbb{R}$ such that for all $s \le s_0$,*

$$\int |v|^2 |y|^{k'} \rho dy \le c_0(k')\alpha^{4-k'} \int \tilde{v}_{1null}^2 \rho dy.$$

Proof : See proof of Lemma C.1 in [MZ98a] (page 187).■

Recalling that $\tilde{v}_1 = \tilde{v}_{1-} + \tilde{v}_{1+} + \tilde{v}_{1null}$, we write on the one hand :

$$\begin{aligned} |\mathcal{E}_1| &\le \int |\tilde{v}_{1+} + \tilde{v}_{1-}| \times |\tilde{v}_1 + \tilde{v}_{1null}||k_2(y)|\rho, \\ &\le c\left(\int |\tilde{v}_{1+} + \tilde{v}_{1-}|^2 \rho\right)^{1/2} \left\{ \left(\int \tilde{v}_1^2 k_2^2(y)\rho\right)^{1/2} + \left(\int \tilde{v}_{1null}^2 k_2^2(y)\rho\right)^{1/2} \right\}. \end{aligned}$$

We have from (2.119) $\left(\int |\tilde{v}_{1+} + \tilde{v}_{1-}|^2\rho\right)^{1/2} = o(\tilde{v}_{12})$ and

$$\left(\int \tilde{v}_1^2 k_2^2(y)\rho\right)^{1/2} + \left(\int \tilde{v}_{1null}^2 k_2^2(y)\rho\right)^{1/2} \le \left(\int |v|^2 k_2^2 \rho\right)^{1/2} + c|\tilde{v}_{12}| \equiv I_1 + I_2.$$

On the other hand, we have :

$$\begin{aligned} \mathcal{E}_3 = \int \frac{1}{2\kappa} \tilde{v}_2^2 k_2(y)\rho(y) &\le c\left(\int \tilde{v}_2^2\rho\right)^{1/2} \left(\int \tilde{v}_2^2 k_2^2 \rho\right)^{1/2}, \\ &\le o(\tilde{v}_{12}) \underbrace{\left(\int |v|^2 k_2^2 \rho\right)^{1/2}}_{I_1}. \end{aligned}$$

To treat I_1, we have from 2.5.4 :

$$\int \tilde{v}_1^2 k_2^2 \rho \le c\int |v|^2\rho + c\int |v|^2|y|^{k'}\rho \le c(\int |v|^2\rho) \le c\tilde{v}_{12}^2.$$

We conclude that $\mathcal{E}_1 = o(\tilde{v}_{12}^2)$ and $\mathcal{E}_3 = o(\tilde{v}_{12}^2)$. We can see easily that $\mathcal{E}_2 = o(\tilde{v}_{12}^2)$, because of Lemma 2.2.3.
It remains to estimate $\mathcal{E}_4$, we consider $\alpha \in (0, \alpha_0)$ and we proceed as in Appendix C from [MZ98a], (page 189). We write for $m = 0$ or $m = 2$:

$$\begin{aligned} \int |v|^3 |y|^m \rho dy &\le \int_{|y|\le\alpha^{-1}} |v|^3|y|^m \rho dy + \int_{|y|\ge\alpha^{-1}} |v|^3 |y|^m \rho dy, \\ &\le \varepsilon\alpha^{-m} \int_{|y|\le\alpha^{-1}} |v|^2\rho dy + CM\alpha^{k'-m} \int_{|y|\ge\alpha^{-1}} |v|^2|y|^{k'}\rho dy, \\ &\le C(\varepsilon\alpha^{-m} + Mc_0(k')\alpha^{4-m}) \int \tilde{v}_{null}^2 \rho dy, \end{aligned}$$

where, we used the fact that $|v| \to 0$ as $s \to -\infty$ in $L^\infty(B(0,\alpha^{-1}))$, $|v(y,s)| \le M$, Lemma 2.5.4 and $\int |v|^2 \rho dy \le \int \tilde{v}_{null}^2 \rho dy$. We can then choose ε and α such that for

$s \leq s_0$, $\int |v|^3|y|^m\rho \leq \varepsilon \int \tilde{v}_{null}^2\rho$ and we obtain $\mathcal{E}_4 = o(\tilde{v}_{12}^2)$.
So finally, we have

$$\frac{d}{ds}(\tilde{v}_{12}) = \frac{(p-\delta^2)}{\kappa}4\tilde{v}_{12}^2 + o(\tilde{v}_{12}^2).$$

Solving the above, we obtain

$$\tilde{v}_{1null} = -\frac{\kappa}{4(p-\delta^2)s}(1+o(1))\big(y^2-2\big).$$

This concludes the proof of Lemma 2.5.3. ■
In order to finish the proof of (iii) of Proposition 2.3.5. We need to refine the estimates of Lemma 2.5.3 to catch the $O(\frac{\log(|s|)}{s^2})$.
Recalling system (2.99) and using (2.118), we obtain,

$$y' \leq -\frac{1}{2}y + c\|v(.,s-s^*)\|_{L^2_\rho}^2 \leq -\frac{1}{2}y + c\frac{1}{s^2}.$$

Then, integrating $(ye^{s/2})' \leq C\dfrac{e^{s/2}}{s^2}$ between $-\infty$ and s, we get $y \leq \frac{C}{s^2}$. Doing the same for $z = \|\tilde{v}_{1+}(.,s)\|_{L^2_\rho}$, we obtain $(ze^{-s/2})' \geq C\dfrac{e^{s/2}}{s^2}$, integrating between s and $s_0 \geq s$, we have $z \leq \dfrac{C}{s^2}$.
Proceeding as in the proof of Proposition 2.5.3, we write :

$$\begin{aligned}
&\frac{d}{ds}(\tilde{v}_{12}) = \frac{p-\delta^2}{2\kappa}\int \tilde{v}_{1null}^2 k_2(y)\rho(y) - \frac{p-\delta^2}{2\kappa}\int(\tilde{v}_{1null}^2 - \tilde{v}_1^2)k_2(y)\rho(y)\\
&+\theta'(s)\int(\delta\tilde{v}_1+\tilde{v}_2)k_2(y)\rho(y) + \int\frac{1}{2\kappa}\tilde{v}_2^2k_2(y)\rho(y) + O\left(\int|v|^3k_2(y)\rho(y)\right),\\
&\equiv \frac{4(p-\delta^2)}{\kappa}\tilde{v}_{12}^2 + \frac{p-\delta^2}{2\kappa}\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4.
\end{aligned} \tag{2.120}$$

Then, we have :

$$\begin{aligned}
|\mathcal{E}_1| &\leq \int|\tilde{v}_{1+}+\tilde{v}_{1-}+\tilde{v}_2|\times|v+\tilde{v}_{1null}||k_2(y)|\rho,\\
&\leq \left(\int|\tilde{v}_{1+}+\tilde{v}_{1-}+\tilde{v}_2|^2\rho\right)^{1/2}\left\{\left(\int v^2k_2^2(y)\rho\right)^{1/2} + \left(\int\tilde{v}_{1null}^2k_2^2(y)\rho\right)^{1/2}\right\},\\
&\leq \varepsilon\left(\int\tilde{v}_{1null}^2\rho\right)^{1/2}\left\{c\left(\int v^4\rho\right)^{1/4} + c\left(\int\tilde{v}_{1null}^2\rho\right)^{1/2}\right\}.
\end{aligned}$$

Using the fact that $\|\tilde{v}_{1null}(.,s)\|_{L^2_\rho} \sim \dfrac{C}{s}$ and (2.118), we have

$$\int v^4\rho \leq c\left(\int v^2(.,s-s^*)\rho\right)^2 \leq \frac{c}{(s-s^*)^2} \leq \frac{c}{s^2}.$$

Thus, $\mathcal{E}_1 \leq \dfrac{C}{s^3}$. Similarly, we obtain $\mathcal{E}_2 \leq \dfrac{C}{|s|^3}$, $\mathcal{E}_3 \leq y^2 \leq \dfrac{C}{s^4}$ and $\mathcal{E}_4 \leq \dfrac{C}{|s|^3}$. Then, we have from (2.120) :

$$\frac{d}{ds}(\tilde{v}_{12}) = \frac{4(p-\delta^2)}{\kappa}\tilde{v}_{12}^2 + O(\frac{1}{s^3}) = \frac{4(p-\delta^2)}{\kappa}\tilde{v}_{12}^2\left(1+O(\frac{1}{s})\right).$$

By integrating, we conclude that :

$$\tilde{v}_{12} = -\frac{\kappa}{4(p-\delta^2)s} + O(\frac{\log|s|}{s^2}).$$

Finally, we get $\|v(.,s) - (1+i\delta)\frac{\kappa}{4(p-\delta^2)s}(y^2-2)\|_{L^2_\rho} = O(\frac{\log|s|}{s^2})$ as $s \to -\infty$. Remains to prove the estimate for $\theta^{'}(s)$ to conclude.

Integrating equation (2.94) with respect to ρdy, we obtain :

$$\theta^{'}(s)\int((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho = \int \tilde{G}_2\rho.$$

On the one hand, we have $((1+\delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) = \kappa + O(\frac{1}{s})$. On the other hand, using (2.96), we get

$$\int \tilde{G}_2\rho = \frac{(1+\delta^2)\delta}{\kappa}\int \tilde{v}_1^2\rho + \frac{(1+\delta^2)}{\kappa}\int \tilde{v}_1\tilde{v}_2\rho,$$

where we have from (iii) of Proposition 2.3.5, $\int \tilde{v}_1\tilde{v}_2\rho = O(\frac{\log|s|}{s^3})$,

$$\begin{aligned}\int \tilde{v}_1^2\rho &= \int \tilde{v}_{12}^2h_2^2\rho + \int\left(\tilde{v}_1^2 - \tilde{v}_{12}^2h_2^2\right)\rho \\ &= 8\tilde{v}_{12}^2 + \int\left((\tilde{v}_1 - \tilde{v}_{12}h_2)\right)(\tilde{v}_1 + \tilde{v}_{12})\rho,\end{aligned}$$

$\tilde{v}_{12} = \frac{\kappa}{4(p-\delta^2)s} + O(\frac{\log|s|}{s^2})$, $\int\left((\tilde{v}_1 - \tilde{v}_{12}h_2)\right)(\tilde{v}_1 + \tilde{v}_{12})\rho \le C\frac{\log|s|}{s^2} \times \frac{1}{s} = C\frac{\log|s|}{s^3}$.

$$\theta^{'}(s) = \frac{(1+\delta^2)\delta}{\kappa}\left(\frac{\kappa}{2(p-\delta^2)}\right)^2\frac{1}{s^2} + O(\frac{\log|s|}{s^3}).$$

Consequently, we obtain the desired estimate for $\theta'(s)$. This concludes the proof of Proposition 2.3.5.■

2.5.2 Equations of Z in Step 4 and 5

Equation of Z in Step 4 : In this part we establish the equation satisfied by Z in Step 4 of the proof of Theorem 1. We note $\phi : \mathbb{C} \to \mathbb{C}$ the function defined by $\phi(x) = |x|^{p-1}x$. If we introduce $\nu = (w - F)$, where F is defined by (2.62), then we see from (2.10) that ν satisfies the following equation for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, such that for $|y| < 4\varepsilon_0\sqrt{-s}$

$$\partial_s\nu = (\mathcal{L}-1)\nu + l(\nu) + B(\nu) + R(y,s), \tag{2.121}$$

where $\mathcal{L}$ is defined in (2.35), $\nu = \nu_1 + i\nu_2$,

$$\begin{aligned}l(\nu) &= (1+i\delta)\left[-\tfrac{\nu}{p-1} + (p-1)|F|^{p-3}F(F_1\nu_1 + F_2\nu_2) + |F|^{p-1}\nu\right],\\ B(\nu) &= (1+i\delta)\left[\phi(F+\nu) - \phi(F) - (p-1)|F|^{p-3}F(F_1\nu_1+F_2\nu_2) - |F|^{p-1}\nu\right],\\ R(y,s) &= -\partial_sF + \Delta F - \tfrac{1}{2}y\cdot\nabla F - (1+i\delta)\tfrac{F}{p-1} + (1+i\delta)|F|^{p-1}F.\end{aligned}$$

Using Taylor's formula and the fact that w and F are bounded for $|y| \le 4\varepsilon_0\sqrt{-s}$, we readily obtain for all $s \le s_0$ and $|y| < 4\varepsilon_0\sqrt{-s}$

$$\begin{aligned} &|B(\nu)| \le C|\nu|^2, \\ &|R(y,s)| \le C\left(\tfrac{|y|^2+1}{s^2} + \chi_{\varepsilon_0}\right), \end{aligned}$$

with χ_{ε_0} defined in (2.64). If we write $\nu = (1+i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$, $B = (1+i\delta)\tilde{B}_1 + i\tilde{B}_2$ and $R = (1+i\delta)\tilde{R}_1 + i\tilde{R}_2$, then we have :

$$\partial_s\tilde{\nu}_1 = \mathcal{L}\tilde{\nu}_1 + l_{1,1}\tilde{\nu}_1 + l_{1,2}\tilde{\nu}_2 + \tilde{B}_1 + \tilde{R}_1 \qquad (2.122)$$

$$\partial_s\tilde{\nu}_2 = (\mathcal{L}-1)\tilde{\nu}_2 + l_{2,2}\tilde{\nu}_2 + l_{2,1}\tilde{\nu}_1 + \tilde{B}_2 + \tilde{R}_2, \qquad (2.123)$$

where

$$\begin{cases} l_{1,1}(y,s) = (1-\delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}({F_1}^2 - \delta^2{F_2}^2) - 1, \\ l_{1,2}(y,s) = -\delta(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}(F_1 - \delta F_2)F_2, \\ l_{2,1}(y,s) = (1+\delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}(F_1 + \delta F_2)F_2, \\ l_{2,2}(y,s) = (1+\delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}{F_2}^2. \end{cases}$$

Proceeding as in the proof of Lemma B.1 from [Zaa98] (page 615), we obtain for all $|y| \le 4\varepsilon_0\sqrt{-s}$

$$|l_{i,j}(y,s)| \le C\min\left[\frac{(1+|y^2|)}{|s|}, 1\right], \text{ for any } i,j \in \{1,2\}.$$

Therefore, we write for $|s|$ large enough and $|y| \le 4\varepsilon_0\sqrt{-s}$:

$$|l_{i,j}(y,s)| \le C\left\{\frac{(1+\varepsilon_0^2|s|)}{|s|} + \chi_{\varepsilon_0}\right\} \le C\left\{2\varepsilon_0^2 + \chi_{\varepsilon_0}\right\}.$$

Now, we multiply (2.122) and (2.123) respectively by $sgn(\tilde{\nu}_1)$ and $sgn(\tilde{\nu}_2)$. Using Kato's inequality, we obtain for $z = |\tilde{\nu}_1| + |\tilde{\nu}_2|$, $|s|$ large enough and $|y| \le 4\varepsilon_0\sqrt{-s}$:

$$\partial_s z - \Delta z + \frac{1}{2}y\cdot\nabla z - (1+\sigma)z \le C\left(z^2 + \frac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right),$$

where we fix ε_0 small enough so that $\sigma = C\varepsilon_0^2 = \frac{1}{100}$.
Now, we consider the cut-off function γ (2.65), we define $Z = z\gamma$ and we obtain for $|s|$ large enough :

$$\begin{aligned} \partial_s Z - \Delta Z + \frac{1}{2}y\cdot\nabla Z - (1+\sigma)Z \;&\le C\left(Z^2 + \tfrac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right) \\ &+ z\left(\partial_s\gamma - \Delta\gamma + \frac{y}{2}\cdot\nabla\gamma\right) - 2\nabla\gamma\cdot\nabla z, \end{aligned}$$

(here, we used the fact that $\gamma z^2 = Z^2 + (\gamma-\gamma^2)z^2 \le Z^2 + C\chi_{\varepsilon_0}$). The last terms in this equation are the cut-off terms. Using the fact that $z\big(\partial_s\gamma - \Delta\gamma + \frac{y}{2}\cdot\nabla\gamma\big) - 2\nabla\gamma\nabla z \le C\chi_{\varepsilon_0} - 2\mathrm{div}(z\nabla\gamma)$, we obtain for $|s|$ large enough :

$$\partial_s Z - \Delta Z + \frac{1}{2}y\cdot\nabla Z - (1+\sigma)Z \le C\left(Z^2 + \frac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right) - 2\mathrm{div}\left((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma\right),$$

which is the desired equation in Lemma 2.3.8.

Equation of Z in Step 5 : In the following, we determine the equation satisfied by Z in Step 4. We note by $\nu = w - f$. We can see from (2.78), that ν satisfies the following equation for all $(y,s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0 e^{-s/2}$

$$\partial_s \nu \;=\; \Delta \nu - \tfrac{1}{2} y \cdot \nabla \nu + l(\nu) + B(\nu) + R(y,s),$$

where

$$\begin{array}{l} l(\nu) = -(1+i\delta)\frac{\nu}{p-1} + (1+i\delta)\left\{(p-1)|f|^{p-3} f(f_1\nu_1 + f_2\nu_2) + |f|^{p-1}\nu\right\}, \\ B(\nu) = (1+i\delta)\left\{|f+\nu|^{p-1}(f+\nu) - |f|^{p-1} f - (p-1)|f|^{p-3} f(f_1\nu_1 + f_2\nu_2) - |f|^{p-1}\nu\right\}, \\ R(y,s) = e^s \Delta G(ye^{s/2}). \end{array}$$

Using a Taylor formula, we prove that for $|s|$ large and $|y| \le 4\varepsilon_0 e^{-s/2}$

$$|B(\nu)| \le C|\nu|^2, |R(y,s)| \le Ce^s + \chi_{\varepsilon_0}(y,s),$$

with χ_{ε_0} is defined by (2.81). If we write $\nu = (1+i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$, $B = (1+i\delta)\tilde{B}_1 + i\tilde{B}_2$ and $R = (1+i\delta)\tilde{R}_1 + i\tilde{R}_2$, then we have :

$$\partial_s \tilde{\nu}_1 \;=\; \mathcal{L}\tilde{\nu}_1 + l_{1,1}\tilde{\nu}_1 + l_{1,2}\tilde{\nu}_2 + \tilde{B}_1 + \tilde{R}_1 \tag{2.124}$$

$$\partial_s \tilde{\tilde{\nu}}_2 \;=\; (\mathcal{L} - 1)\tilde{\tilde{\nu}}_2 + l_{2,2}\tilde{\nu}_2 + l_{2,1}\tilde{\nu}_1 + \tilde{B}_2 + \tilde{R}_2, \tag{2.125}$$

where

$$\left\{\begin{array}{lcl} l_{1,1}(y,s) &=& (1-\delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}({f_1}^2 - \delta^2 {f_2}^2) - 1, \\ l_{1,2}(y,s) &=& -\delta(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}(f_1 - \delta f_2) f_2, \\ l_{2,1}(y,s) &=& (1+\delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}(f_1 + \delta f_2) f_2, \\ l_{2,2}(y,s) &=& (1+\delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}{f_2}^2. \end{array}\right.$$

Proceeding as in the proof of Lemma B.1 from [Zaa98] (page 615), we obtain for $|y|e^{s/2} \le 4\varepsilon_0$ and s large

$$|l_{i,j}(y,s)| \le C \min\left[|y|e^{s/2}, 1\right], \text{ for any } i,j \in \{1,2\}.$$

If we consider χ_{ε_0} defined in (2.81), then, we write for $|s|$ large and $|y| \le 4\varepsilon_0 e^{-s/2}$:

$$|l_{i,j}| \le C\left\{|y|e^{s/2} + \chi_{\varepsilon_0}\right\} \le C\left\{\varepsilon_0 + \chi_{\varepsilon_0}\right\}.$$

Now, we multiply (2.124) and (2.125) respectively by $sgn(\tilde{\nu}_1)$ and $sgn(\tilde{\nu}_2)$. Using Kato's inequality, we obtain for $z = |\tilde{\nu}_1| + |\tilde{\nu}_2|$, $|s|$ large enough and $|y|e^s \le 4\varepsilon_0$,

$$\partial_s z - \Delta z + \frac{1}{2} y \cdot \nabla z - (1+\sigma) z \le C\left(z^2 + e^s + \chi_{\varepsilon_0}\right),$$

where $\sigma = C\varepsilon_0 = \frac{1}{100}$. Now, we consider the cut-off function γ, we define $Z = z\gamma$ and we obtain for $|s|$ large :

$$\begin{array}{l} \partial_s Z - \Delta Z + \dfrac{1}{2} y \cdot \nabla Z - (1+\sigma) Z \\ \le C\left(Z^2 + e^s + \chi_{\varepsilon_0}\right) - z\left(\partial_s \gamma - \Delta \gamma + \dfrac{y}{2} \cdot \nabla \gamma\right) + 2\nabla\gamma\nabla z. \end{array}$$

The last terms in this equation are the cut-off terms. Using $z\big(\partial_s\gamma-\Delta\gamma+\frac{y}{2}\cdot\nabla\gamma\big)-2\nabla\gamma\nabla z \leq C\chi_{\varepsilon_0}+2\text{div}(z\nabla\gamma)$, we obtain for $|s|$ large :

$$\partial_s Z-\Delta Z+\frac{1}{2}y\cdot\nabla Z-(1+\sigma)Z\leq C\left(Z^2+e^s+\chi_{\varepsilon_0}\right)-2\text{div}\left((|\tilde{\nu}_1|+|\tilde{\nu}_2|)\nabla\gamma\right),$$

which is the desired equation in (2.80).

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Chapitre 3

A Liouville theorem for a heat equation and applications for quenching

In *preparation*

A Liouville theorem for a heat equation and applications for quenching

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We prove a Liouville Theorem for a semilinear heat equation with absorption term in one dimension. We also give some uniform estimates for quenching solutions.

Mathematical Subject classification : 35K05, 35K55, 74H35.
Keywords : Quenching, Liouville theorem, heat equation.

3.1 Introduction

This paper is concerned with quenching solutions of the nonlinear heat equation

$$\begin{cases} \partial_t u &= \partial_x^2 u - \dfrac{1}{u^\beta} \text{ in } \mathbb{R}\times[0,T), \\ u(x,0) &= u_0(x) > 0 \text{ for } x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $\beta \geq 3$ and

$$u_0, \quad \frac{1}{u_0} \in L^\infty(\mathbb{R}). \tag{3.2}$$

For a discussion about the limitation of our work to the case $\beta \geq 3$, see the third remark following Theorem 4 below.
We say that $u(t)$ quenches in finite time T if u exists for $t \in [0,T)$ and

$$\lim_{t\to T} \inf_{x\in\mathbb{R}} u(x,t) = 0. \tag{3.3}$$

Hereafter we consider a solution u of (3.1) which quenches at finite time T. A point a is said to be a quenching point if there is a sequence $\{(a_n,t_n)\}$ such that $a_n \to a$, $t_n \to T$ and $u(a_n,t_n) \to 0$ as $n \to \infty$.
Quenching phenomena play an important role in the theory of plasma physics, combustion, detonation and ecology. It is also important in differential geometry and in the related environmental researches (see for example Altschuler, Angenent and Giga [AAG95], Deng [Den92], Dziuk and Kawohl [DK91] and Galaktionov, Gerbi and Vazquez [GGV01] and the references cited therein). The study of the quenching problem (3.1) was initiated by Kawarada [Kaw75]. Then, many authors addressed questions about existence and the qualitative behavior of quenching solutions (see Deng and Levine [DL89], Fila and Kawohl [FK92a], Fila, Kawohl and Levine [FKBL92], Levine [Lev93] and Dávila and Montenegro [DM05]). Recently, equation (3.1) was numerically studied by Liang, Lin and Tan [LLT07]. The quenching rate as $t \to T$ of the solution u of (3.1) ($\Omega = [-1,1]$) near a quenching

point is among the important issues. It was obtained by Guo [Guo90], [Guo91a] and for the higher-dimensional radial problem in [Guo91b] :

If we define $U_T(t) = (\beta+1)^{\frac{1}{\beta+1}}(T-t)^{\frac{1}{\beta+1}}$ *the solution of* $U_T' = -\frac{1}{U_T^\beta}$ *and* $U_T(T) = 0$*, then we have*

$$U_T(t) \geq \inf_{-1<x<1} u(x,t) \geq C_0 U_T(t), \text{ for some positive constant } C_0. \tag{3.4}$$

We note that the upper bound is much easier to obtain and follows from the maximum principle.

In [FG93], Filippas and Guo were able to obtain a precise description of the profile of the solution in a neighborhood of the quenching point. They proved the following :

Let u *be a solution of (3.1) which quenches at the point* a *at time* T*. Moreover, we assume that* $u_0'' - \frac{1}{u_0^\beta} \leq 0$ *and that* u_0 *has a single minimum. Then, we have that*

$$\begin{cases} u(a,t) & \rightarrow & 0, \\ u(x,t) & \rightarrow & u^*(x) \text{ as } t \rightarrow T \text{ if } x \neq a, \\ u^* & = & \Big[\frac{(\beta+1)^2}{8\beta}\Big]^{\frac{1}{(\beta+1)}}\Big(\frac{|x-a|^2}{|\log|x-a||}\Big)^{\frac{1}{(\beta+1)}}(1+o(1)), \end{cases} \tag{3.5}$$

as $|x-a| \rightarrow 0$.

However, the result of Filippas and Guo is not uniform with respect to the quenching point or initial data. We aim in this paper in obtaining uniform estimates on $u(t)$ at or near the singularity, that is as $t \rightarrow T$. In order to do so, we introduce for each $a \in \mathbb{R}$ the following similarity variables :

$$y = \frac{x-a}{\sqrt{T-t}},\ s = -\log(T-t),\ w_a(y,s) = (T-t)^{-\frac{1}{\beta+1}}u(x,t). \tag{3.6}$$

The function $w_a (= w)$ satisfies for all $s \geq -\log T$, and $y \in \mathbb{R}$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \cdot \partial_y w + \frac{w}{\beta+1} - w^{-\beta}, \tag{3.7}$$

The study of $u(t)$ near (a,T), where a is a quenching point, is equivalent to the study of the long time behavior of w_a.
If we define

$$v = \frac{1}{u}, \tag{3.8}$$

then, we can see easily from (3.1) and (3.3), that v satisfies the following equation :

$$\partial_t v = \partial_x^2 v - 2\frac{(\partial_x v)^2}{v} + v^{2+\beta} \text{ in } \mathbb{R} \times [0,T), \tag{3.9}$$

We note that u quenches at time T if and only if v blows up at time T, and that a is a quenching point for u if and only if a is a blow-up point for v.
As we did for u, we introduce for v the following function :

$$z_a(y,s) = (T-t)^{\frac{1}{\beta+1}} v(x,t), \text{ where } y \text{ and } s \text{ are defined as in (3.6).} \tag{3.10}$$

Note from (3.6) and (3.8) that $z_a = \frac{1}{w_a}$. From (3.9) and (3.10), see that $z_a (= z)$ satisfies for all $s \geq -\log T$, and $y \in \mathbb{R}$,

$$\partial_s z = \partial_y^2 z - \frac{1}{2} y \cdot \partial_y z - 2\frac{(\partial_y z)^2}{z} - \frac{z}{\beta+1} + z^{2+\beta}. \tag{3.11}$$

We introduce also the Hilbert space

$$L^2_\rho = \{g \in L^2_{loc}(\mathbb{R}^N, \mathbb{C}), \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy < +\infty\} \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

If g depends only on the variable $y \in \mathbb{R}^N$, we use the notation

$$\|g\|^2_{L^2_\rho} = \int_{\mathbb{R}^N} |g(y)|^2 e^{-\frac{|y|^2}{4}} dy.$$

If g depends only on $(y,s) \in \mathbb{R}^N \times \mathbb{R}$, we use the notation

$$\|g(.,s)\|^2_{L^2_\rho} = \int_{\mathbb{R}^N} |g(y,s)|^2 e^{-\frac{|y|^2}{4}} dy.$$

3.1.1 A Liouville Theorem

Theorem 4. *(A Liouville Theorem for equation (3.7)) Assume that $\beta \geq 3$. We consider w, a global nonnegative continuous solution of (3.7), satisfying*

$$|\partial_y w(y,s)| + \frac{1}{w(y,s)} \leq M, \text{ for all } (y,s) \in \mathbb{R} \times \mathbb{R}, \text{ where } M > 0. \tag{3.12}$$

Then either
(i) $w \equiv \kappa$, or
(ii) there exists $s_0 \in \mathbb{R}$ such that for all$(y,s) \in \mathbb{R} \times \mathbb{R}$, $w(y,s) = \varphi(s - s_0)$, where

$$\varphi(s) = \kappa(1 + e^s)^{\frac{1}{\beta+1}} \text{ and } \kappa = (\beta+1)^{1/(\beta+1)}. \tag{3.13}$$

Remark : Note that φ is a solution of (3.7) independent of y and satisfies

$$\varphi' = \frac{\varphi}{1+\beta} - \varphi^{-\beta}, \quad \varphi(-\infty) = \kappa, \quad \varphi(+\infty) = +\infty. \tag{3.14}$$

Remark : The boundedness of $\partial_y w$ in (3.12) is a natural hypothesis, since it appears in the physical case of the reconnection of a vortex with the boundary in a type II superconductor. See [MZ97].
Remark : From (3.13), one may wander why our result dosen't hold for all $\beta > -1$. In fact, the limitation to the case $\beta \geq 3$ comes from the fact that when $-1 < \beta < 3$, we don't have a classification of stationary solution of (3.7) (we know however that in addition to (3.13), there are slow orbits ; for more details see Guo [Guo91a] and [Guo91b]).
Theorem 4 has an equivalent formulation for solutions for (3.1) :

Corollary 5. *(A Liouville Theorem for equation (3.1).) Assume that u is a positive continuous solution of (3.1) defined for $(x,t) \in \mathbb{R} \times (-\infty, T)$. Assume in addition that for some $M > 0$*

$$u(x,t) \geq \frac{1}{M}(T-t)^{\frac{1}{(\beta+1)}} \text{ and } |\partial_x u(x,t)| \leq M\,(T-t)^{\frac{1}{\beta+1}-\frac{1}{2}} \text{ for all } (x,t) \in \mathbb{R} \times (-\infty, T).$$

Then, there exists $T_0 \geq T$ such that for all $(x,t) \in \mathbb{R} \times (-\infty, T)$, $u(x,t) = \kappa(T_0 - t)^{\frac{1}{(\beta+1)}}$.

3.1.2 Application to quenching

We note that in the blow-up recent litterature ([MZ00], [MM00] and [MZ08]), Liouville Theorems have important applications to blow-up. We think that in the quenching problem we can get similar results. If we proceed as [MZ00] and [NZ08], then we can derive the following estimates, that will proved in a future work.

- *(ODE-type behavior) Assume that $u(t)$ is a nonnegative solution of equation (3.1) that quenches in finite time $T > 0$. Then, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that, for all $t \in [\frac{T}{2}, T)$ and $x \in \mathbb{R}$,*

$$\left| \frac{\partial u}{\partial t} - u^{-\beta} \right| \leq \varepsilon |u|^{-\beta} + C_\varepsilon. \tag{3.15}$$

- *(Uniform bound on $u(t)$ on quenching time) Consider $u(t)$ a quenching solution of equation (3.1) that quenches at time T. In addition, assume that $u_0'' - \frac{1}{u_0^\beta} \leq 0$. Then,*

$$\inf_{x \in \mathbb{R}} |u(x,t)| \to U_T(t) = (\beta+1)^{\frac{1}{\beta+1}}(T-t)^{\frac{1}{\beta+1}} \text{ as } t \to T,$$

and

$$(T-t)^{-\frac{1}{\beta+1}+\frac{1}{2}} \|\partial_x u(.,t)\|_{L^\infty(\mathbb{R})} + (T-t)^{-\frac{1}{\beta+1}+1} \|\partial_x^2 u(.,t)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } t \to T.$$

Equivalently, for any $a \in \mathbb{R}$,

$$\inf_{y \in \mathbb{R}} |w_a(y,s)| \to \kappa \text{ as } s \to +\infty \text{ and } \|\partial_y w(.,s)\|_{L^\infty} + \|\partial_y^2 w(.,s)\|_{L^\infty} \to 0 \text{ as } s \to +\infty,$$

where $\kappa = (1+\beta)^{\frac{1}{\beta+1}}$.

3.1.3 Strategy of the proof of the Liouville theorem

Our method is inspired by the one of Merle and Zaag [MZ98a] and [MZ00] developed for the semilinear heat equation

$$\partial_t u \; = \; \Delta u + |u|^{p-1} u \tag{3.16}$$

where

$$u : (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R} \text{ and } p > 1, \quad p < \frac{N+2}{N-2} \text{ if } N \geq 3.$$

However, our contribution is not a simple adaptation of the proof of [MZ98a] and [MZ00]. In fact, in these papers the authors strongly rely on two blow-up criterion for the selfsimilar version of (3.16)

$$w_s = \Delta w - \frac{1}{2} y\partial_y w - \frac{w}{p-1} + |w|^{p-1}w. \tag{3.17}$$

- Criterion 1 : For nonnegative solution of (3.17) such that

$$\int_{\mathbb{R}^N} w(y, s_0)\rho(y)dy > (p-1)^{-\frac{1}{p-1}},$$

  for some $s_0 \in \mathbb{R}$, then w blows up in some finite time $s > s_0$.
- Criterion 2 : (with no sign condition) Assume that w is a solution of (3.17) such that

$$\mathcal{E}(w(s_0)) \leq \frac{(p-1)}{2(p+1)} \Big(\int_{\mathbb{R}^N} |w(y,s_0)|^2 \rho(y)dy \Big)^{\frac{p+1}{2}},$$

  for some $s_0 \in \mathbb{R}$, where $\mathcal{E}$ is the following Lyapunov functional defined by

$$\mathcal{E}(w) = \int_{\mathbb{R}} \Big(\frac{1}{2}|\partial_y w|^2 + \frac{|w|^2}{2(p-1)} - \frac{|w|^{p+1}}{p+1} \Big) \rho(y)dy \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

  Then, w blows up in finite time $s > s_0$.

In [NZ08], we proved a Liouville theorem for the following equation with no gradient structure

$$\partial_t u = \Delta u + (1 + i\delta)|u|^{p-1}u \tag{3.18}$$

where $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$. Note that in [Zaa01], Zaag obtained a Liouville theorem for the non gradient structure system

$$\begin{cases} \partial_t u = \Delta u + v^p, & \partial_t v = \Delta v + u^q \\ u(.,0) = u_0, & v(.,0) = v_0, \end{cases} \tag{3.19}$$

where $p - q$ is small. There, he adapts the proof of [MZ98a] and uses an infinite time blow-up criterion in similarity variables.
In our quenching problem, there is no way to find any equivalent blow-up criterion. We need new ideas which make the originality of our paper. From this point of view the difficulty is similar to the case of the complex valued equation (3.18) treated in [NZ08]. We proceed in five parts.

- Part 1, we show that $z \left(= \dfrac{1}{w} \right)$ has a limit $z_{\pm\infty}$ as $s \to \pm\infty$, where $z_{\pm\infty}$ is a critical point of the stationary version of (3.11) or 0. That is, $z_{\pm\infty} \equiv \kappa^{-1}$, or $z_{\pm\infty} \equiv 0$. Then, we rule out the case where $z_{-\infty} \equiv 0$. The following parts are dedicated to the non trivial case where $z_{-\infty} = \kappa^{-1}$.
- in part 2, we investigate the linear problem (in w) around κ as $s \to -\infty$ and show that w behaves at most in three ways :
  - (i) $w(y,s) = \kappa + C_0 e^s + o(e^s)$ as $s \to -\infty$, for some constant $C_0 \in \mathbb{R}$.
  - (ii) $w(y,s) = \kappa + C_1 e^{s/2} + o(e^{s/2})$ as $s \to -\infty$, for some constant $C_1 \in \mathbb{R} \setminus \{0\}$.

– (iii) $w(y,s) = \kappa + \frac{\kappa}{2\beta s}\left(\frac{1}{2}y^2 - 1\right) + o\left(\frac{1}{s}\right)$ as $s \to -\infty$.

In these cases, convergence take place in $L^\infty([-R,R])$ for any $R > 0$ and in L^2_ρ.

- In part 3, we show that (i) corresponds to $w(y,s) = \varphi(s - s_0)$ or $w(y,s) = \kappa$, for some $s_0 \in \mathbb{R}$, where φ is defined by (3.14).
- In part 4 and 5, we rule out cases (ii) and (iii). In [MZ98a] and [MZ00], the authors shows that for some $a_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, $w_{a_0}(y,s) = w(y + a_0 e^{s_0}, s_0)$ satisfies one of the blow-up criteria stated in page 86, which contradicts the fact that w exists for all $s \in \mathbb{R}$. In our case we haven't any blow-up criterion. It turns out that this is the major difficulty in our paper, as in [NZ08] for equation (3.18). Following [NZ08], we will use a geometrical method where the key idea is to extend the convergence stated in (ii) and (iii) from compact sets to larger zones, so that we find the *profile* of w. It appears that in both cases, for larger $|y|$, this profile becomes strictly inferior to $\frac{1}{M}$, where M is defined in (3.12), which is a contradiction. The originality of our paper is based on Velázquez's work in [Vel92], where he extends the convergence from compact sets to larger sets to find the profile for solutions of (3.17). Note that the fact that w is not in L^∞ makes it delicate to use estimate of [Vel92]. More precisely, we obtain the following profiles
  – If case (ii) holds, then

$$\lim_{s\to-\infty} \sup_{|y|\le Re^{-s/2}} \left| w(y,s) - ((\beta+1) - C_1\kappa^{-\beta} y e^{s/2})^{\frac{1}{\beta+1}} \right| = 0$$

  with $0 < R < \frac{(\beta+1)}{C_1\kappa^{-\beta}}$ and $C_1 \neq 0$.

  – If case (iii) holds, then

$$\lim_{s\to-\infty} \sup_{|y|\le\sqrt{-s}R} \left| w(y,s) - \left((\beta+1) + \frac{(\beta+1)^2}{4\beta}\frac{y^2}{s}\right)^{\frac{1}{\beta+1}} \right|$$

  with $0 < R < \sqrt{\frac{4\beta}{(\beta+1)}}$.

  Then using condition (3.12), case (ii) and (iii) are ruled out, which ends the proof of our Liouville Theorem.

3.2 Proof of the Liouville Theorem for equation (3.7)

We assume that $\beta \geq 3$ and consider $w(y,s)$ a nonnegative, global solution of (3.7) satisfying (3.12), defined for all $(y,s) \in \mathbb{R}\times\mathbb{R}$. Introducing $z = \frac{1}{w}$, we know that z satisfies (3.11) on $\mathbb{R}^2$.
Our goal is to show that w depends only on the variable s.

3.2.1 Part 1 : Behavior of w as $s \to \pm\infty$

The main results of this part are consequences of parabolic estimates and the gradient structure of equation (3.7). Let us recall them.

Lemma 3.2.1. *(Parabolic estimates) There exists $M_0 > 0$ such that for all $(y,s) \in \mathbb{R}\times\mathbb{R}$:*
(i) $|\partial_y w(y,s)| + \dfrac{1}{w(y,s)} \leq M_0$ *and* $w(y,s) \leq w(0,s) + M_0|y|$,
(ii) $|z(y,s)| + |\partial_y z(y,s)| + |\partial_y^2 z(y,s)| \leq M_0$ *and* $|\partial_s z| \leq M_0(1+|y|)$.
(iii) $|\partial_s w(y,s)| \leq M_0(1+|y|)(w(0,s)+|y|)^2$.
(iv) For all $R > 0$, z, $\partial_y z(y,s)$, $\partial_y^2 z$ and $\partial_s z$ are bounded in $C^{0,\alpha}([-R,R]^2)$ for some $\alpha \in (0,1)$, where

$$C^{0,\alpha}([-R,R]^2) = \left\{ \psi \in L^\infty[(-R,R)^2] \mid \sup_{(\xi,\tau),(\xi',\tau')\in[-R,R]^2} \frac{|\psi(\xi,\tau)-\psi(\xi',\tau')|}{(|\xi-\xi'|+|\tau-\tau'|^{1/2})^\alpha} < \infty \right\}$$

Proof : (i) See (3.12).
(ii) See the proof of Lemma 3.2 from [Guo90].
(iii) If we write $\partial_s w = \partial_s z/z^2 = \partial_s z w^2$, then we get the result by (i) an (ii).
(iv) It follows from classical Schauder estimates. ■

Lemma 3.2.2. *(Gradient Structure)*
(i) (Gradient Structure for equation (3.7)). If we introduce

$$\boldsymbol{E}(w(s)) = \frac{1}{2}\int_{\mathbb{R}} \partial_y w^2 \rho dy - \frac{1}{2(\beta+1)}\int_{\mathbb{R}} w^2 \rho dy - \frac{1}{\beta-1}\int_{\mathbb{R}} w^{-(\beta-1)}\rho dy \text{ with } \rho(y) = \frac{e^{-\frac{y^2}{4}}}{(4\pi)^{\frac{1}{2}}}, \tag{3.20}$$

then, for all $s_1 < s_2 \in \mathbb{R}$, such that

$$0 < w(y,s) \leq c_1(s)(1+|y|) \text{ for all } s_1 \leq s \leq s_2 \text{ (with } c_1(s) > 0\text{)},$$

we have

$$\int_{s_1}^{s_2}\int_{\mathbb{R}} (w_s)^2 \rho dy = \boldsymbol{E}(w(s_1)) - \boldsymbol{E}(w(s_2)) \tag{3.21}$$

(ii) (Gradient Structure for equation (3.11)).If we define for each z solution of (3.11)

$$E(z(s)) = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial_y z^2}{z^4}\rho dy - \frac{1}{2(\beta+1)}\int_{\mathbb{R}} \frac{1}{z^2}\rho dy - \frac{1}{\beta-1}\int_{\mathbb{R}} z^{\beta-1}\rho dy, \tag{3.22}$$

under the same conditions in (i), we obtain :

$$\int_a^b \int_{\mathbb{R}} \left(\frac{z_s}{z^2}\right)^2 \rho dy = E(z(a)) - E(z(b)), \text{ for any real } a < b. \tag{3.23}$$

Proof :
(i) One may multiply equation (3.11) by $\partial_s w\rho$ and integrate over the ball $B_R = B(O,R)$ with $R > 0$. Then using Lemma 3.2.1 and the Lebesgue's theorem yields the result. (see Proposition 3 from Giga and Kohn [GK85] for more details).
(ii) Since $E(z) = \mathbf{E}(\frac{1}{z})$, this immediately follows from (i).■

We now give the limits of z as $s \to \pm\infty$ in the following :

Proposition 3.2.3. *(Limit of z as $s \to \pm\infty$) The limit $z_{+\infty}(y) = \lim_{s\to+\infty} z(y,s)$ exists and equals 0 or κ^{-1}. The convergence is uniform on every compact subset of $\mathbb{R}$. The corresponding statements also hold for the limit $z_{-\infty}(y) = \lim_{s\to-\infty} z(y,s)$.*

Proof : It follows from the following :

Lemma 3.2.4. *Consider any increasing (respectively decreasing) sequence such that $s_j \to \pm\infty$ as $j \to \infty$. Then :*
(i) there is a subsequence (still denoted by (s_j)) such that

$$z(y, s+s_j) \to l \text{ in } C^{2,\alpha}([-R,R]) \text{ for all } R>0, \text{where } l = 0 \text{ or } \kappa^{-1}.$$

(ii) If $l=0$, then $E(z(s_j)) \to -\infty$, if $l = \kappa^{-1}$, then $E(z(s_j)) \to E(\kappa^{-1})$.

Indeed, from Lemma 3.2.4, in order to get the conclusion of Proposition 3.2.3, it is enough to show that the limit in Lemma 3.2.4 is independent of the choice of the sequence. We consider the case $s \to +\infty$, the other case being similar. Suppose that (s_j) and $(\overline{s}_j)$ both tend to infinity. Up to extracting subsequence, we assume that for all $j \in \mathbb{N}$, $\quad s_j > \overline{s}_j$ and proceeding by contradiction, we assume that

$$z_j(y,s) = z(y, s+s_j) \to \kappa^{-1} \text{ and } \overline{z}_j(y,s) = z(y, s+\overline{s_j}) \to 0 \text{ as } j \to \infty.$$

By (ii) of Lemma 3.2.4, we have

$$E(z_j(s)) \to E(\kappa^{-1}) \text{ and } E(\overline{z}_j(s)) \to -\infty,$$

hence, for j large enough, we have

$$s_j > \overline{s}_j, \ E(z(s_j)) > E(\kappa^{-1}) - 1 > E(z(\overline{s}_j)),$$

which contradicts the monotonicity of E. Thus the limit in Lemma 3.2.4 is independent of the choice of the sequence and the whole function $z(y,s)$ converges as $s \to \infty$. It remains to prove Lemma 3.2.4 to finish the proof.

Proof of Lemma 3.2.4 :
(i) We only present the case $s_j \to +\infty$, the analysis for $s_j \to -\infty$ is the same.
Let us recall that for all (y,s), $z(y,s)$ satisfies (3.11), which is :

$$\partial_s z = \partial_y^2 z - \frac{1}{2} y \partial_y z - 2\frac{(\partial_y z)^2}{z} - \frac{z}{\beta+1} + z^{2+\beta}. \tag{3.24}$$

From condition (3.12), we get $0 \le \dfrac{(\partial_y z)^2}{z} = \dfrac{(\partial_y w)^2}{w^2} z \le Cz$ and $0 \le z^{2+\beta} \le Cz$, hence we obtain :

$$\mathcal{L}_0 z - Cz \le \partial_s z \le \mathcal{L}_0 z + Cz \text{ where } \mathcal{L}_0 z = \partial_y^2 z - \frac{1}{2} y \partial_y z. \tag{3.25}$$

Using the semigroup $S_0(\tau)$ associated to $\mathcal{L}_0$:

$$S_0(\tau)\phi(y) = \frac{1}{(4\pi(1-e^{-\tau}))^{1/2}} \int_{\mathbb{R}} \exp\Big(- \frac{(ye^{-\frac{\tau}{2}} - \lambda)}{4(1-e^{-\tau})}\Big) \phi(\lambda) d\lambda, \tag{3.26}$$

we see that for all $s \geq s'$,

$$e^{-C(s-s')} S_0(s-s') z(.,s') \leq \|z(.,s)\|_{L^\infty} \leq e^{C(s-s')} \|z(.,s')\|_{L^\infty}. \tag{3.27}$$

Let (s_j) be a sequence tending to $+\infty$, and let $z_j(y,s) = z(y, s+s_j)$.
From Lemma 3.2.1, up to extracting a subsequence (still denoted by (s_j)), z_j converges to $\tilde{z}_{+\infty}$ in $C^{2,1}((-R,R)^2)$ for any $R > 0$ and we obtain that $\tilde{z}_{+\infty}$ satisfy (3.25).

In the following, we will prove that the limit $\tilde{z}_{+\infty}$ is either 0 or κ^{-1}. We have to consider two cases. .

Case 1 : There exists $(y_0, s_0) \in \mathbb{R}^2$ *such that* $\tilde{z}_{+\infty}(y_0, s_0) = 0$
We claim first that

$$\forall y \in \mathbb{R}, \quad \tilde{z}_{+\infty}(y, s_0) = 0. \tag{3.28}$$

Indeed, if for some $y_1 \in \mathbb{R}$, we have $\tilde{z}_{+\infty}(y_1, s_0) > 0$, then since we have from (3.12)

$$\frac{1}{z_j(y_0, s_0)} \leq \frac{1}{z_j(y_1, s_0)} + M|y_1 - y_0|,$$

letting $j \to \infty$, we get a contradiction. Thus (3.28) holds. Using (3.27), we conclude that $z \equiv 0$ on $\mathbb{R}^2$ (indeed, by (3.27) for $s \geq s_0$, $z(.,s) \equiv 0$, and for $s \leq s_0$, $S_0(s_0 - s) z(.,s) \equiv 0$, hence $z(.,s) \equiv 0$).

Case 2 : For all $(y,s) \in \mathbb{R}^2$, $\tilde{z}_{+\infty}(y,s) > 0$
Let us introduce $w_j(y,s) = w(y, s+s_j)$. In this case,

$$w_j(y,s) = 1/z_j(y,s) \to \tilde{w}_{+\infty} = 1/\tilde{z}_{+\infty}(y,s) \text{ in } C^2((-R,R)^2, \text{ for any } R > 0. \tag{3.29}$$

Since z_j and w_j are solutions of (3.24) and (3.7) respectively, the same holds for $\tilde{z}_{+\infty}$ and $\tilde{w}_{+\infty}$ respectively. Our goal in this step is to prove that $\tilde{z}_{+\infty}(y,s) = \kappa^{-1}$. First, we prove that $\tilde{z}_{+\infty}$ is independent of s, then we conclude using the result of Guo concerning the stationary global solution of (3.7). We will proceed as in Proposition 4, page 308 in [GK85].
We claim the following :

Claim 3.2.5. *(i) For all* $(y,s) \in \mathbb{R}^2$

$$\begin{aligned} &|\partial_y w_j(y,s)| + |1/w_j(.,s)| \leq C_1, \quad |w_j(y,s)| \leq C_1(|y| + \tilde{w}_{+\infty}(0,s)) \\ &\text{and } |\partial_s w_j(y,s)| \leq C_1(1+|y|)\left(|y| + \tilde{w}_{+\infty}(0,s)\right)^2, \end{aligned}$$

where $C_1 > 0$ *independent of* j *and* $\tilde{w}_{+\infty}$ *is a solution of (3.7).*

(ii) For all $(y,s) \in \mathbb{R}^2$,

$$\begin{aligned} &|\partial_y \tilde{w}_{+\infty}(y,s)| + |1/\tilde{w}_{+\infty}(y,s)| \leq C_1, \quad |\tilde{w}_{+\infty}(y,s)| \leq C_1(|y| + \tilde{w}_{+\infty}(0,s)) \\ &\text{and } |\partial_s \tilde{w}_{\infty}(y,s)| \leq C_1(1+|y|)\left(|y| + \tilde{w}_{+\infty}(0,s)\right)^2. \end{aligned} \tag{3.30}$$

Remark : To make the notation legible, we will note the partial derivative in time and space of the sequence w_j (or z_j) and the limit $\tilde{w}_{+\infty}$ (or $\tilde{z}_{+\infty}$) respectively $\partial_s w_j$, $\partial_y w_j$ and $\partial_s \tilde{w}_{+\infty}$, $\partial_y \tilde{w}_{+\infty}$.

Proof of Claim 3.2.5 : (i) Integrating the inequality $|\partial_y w_j(y,s)| \leq M$ (by (i) of Lemma 3.2.1) in space between 0 and y, we obtain $|w_j(y,s)| \leq M|y| + w_j(0,s)$. Then using the fact that

$$w_j(0,s) \to \tilde{w}_{+\infty}(0,s)(= 1/\tilde{z}_{+\infty}(0,s)) < +\infty, \text{ as } j \to \infty,$$

we get the second inequality of (i) in Claim 3.2.5. Using Lemma 3.2.1 and the identity

$$\partial_s w_j(y,s) = -\partial_s z_j(y,s)/z_j(y,s)^2 = -\partial_s z_j(y,s) w_j(y,s)^2,$$

we obtain the third inequality in (i).
(ii) Using (i) and the convergence (3.29), the result comes immediately.■

Up to extracting a subsequence, we can assume that $s_{j+1} - s_j \to \infty$. Therefore, using (3.21) with $w = w_j$, $a = m$, $b = m + s_{j+1} - s_j$, where $m \in \mathbb{Z}$, we get :

$$\begin{aligned} \int_m^{s_{i+1}-s_j+m} \int_{\mathbb{R}} (\partial_s w_j)^2 \rho dy &= \mathbf{E}(w_j(m)) - \mathbf{E}(w_j(m + s_{j+1} - s_j)) \\ &= \mathbf{E}(w_j(m)) - \mathbf{E}(w_{j+1}(m)). \end{aligned} \tag{3.31}$$

Using (i) and (ii) of Claim 3.2.5 together with Lesbegue's Theorem we have

$$\int_{\mathbb{R}} \partial_y w_j^2 \rho dy \to \int_{\mathbb{R}} \partial_y \tilde{w}_{+\infty}^2 \rho dy, \qquad \int_{\mathbb{R}} w_j^2 \rho dy \to \int_{\mathbb{R}} \tilde{w}_{+\infty}^2 \rho dy,$$
$$\text{and } \int_{\mathbb{R}} w_j^{-(\beta-1)} \rho dy \to \int_{\mathbb{R}} \tilde{w}_{+\infty}^{-(\beta-1)} \rho dy \quad \text{as } j \to \infty.$$

Therefore, $\mathbf{E}(w_j(m)) \to \mathbf{E}(\tilde{w}_{+\infty}(m))$ as $j \to \infty$ (well defined by (iii) of Claim 3.2.5). In particular, the right-hand side of (3.31) tends to zero. Since $s_{j+1} - s_j \to +\infty$, it follows that for every integers $m < M$

$$\text{on the one hand } \lim_{j\to\infty} \int_m^M \int_{\mathbb{R}} |\partial_s w_j|^2 \rho dy ds = 0, \tag{3.32}$$

On the other hand, by the third inequality in (i) of Claim 3.2.5 and the continuity of $\tilde{w}_{+\infty}(0,s)$, we obtain

$$\forall (y,s) \in \mathbb{R} \times (m,M) \quad |\partial_s w_j(y,s)| \leq C(m,M)(1+|y|)^3.$$

Since $\partial_s w_j$ converges simply to $\partial_s \tilde{w}_{+\infty}$ by (iii) of Claim 3.2.5, we conclude that :

$$\int_m^M \int_{\mathbb{R}} |\partial_s w_j|^2 \rho dy ds \to \int_m^M \int_{\mathbb{R}} |\partial_s \tilde{w}_{+\infty}|^2 \rho dy ds. \tag{3.33}$$

Using (3.32) and (3.33), we get

$$\int_m^M \int_{\mathbb{R}} |\partial_s \tilde{w}_{+\infty}|^2 \rho dy ds = 0.$$

Using the fact that m and M are arbitrary, we conclude that $\partial_s \tilde{w}_{+\infty} = 0$ on $\mathbb{R}^2$ and $\tilde{w}_{+\infty}(y,s) = \tilde{w}_{+\infty}(y)$ is independent of s. Since $\tilde{w}_{+\infty}$ is a solution of (3.7), it follows that $\tilde{w}_{+\infty}$ solves the following stationary equation

$$\forall y \in \mathbb{R}, \;\; 0 = w'' - \frac{1}{2} y w' + \frac{1}{\beta+1} w - w^{-\beta}. \tag{3.34}$$

We recall that by (ii) of Claim 3.2.5 we have $\tilde{w}_{+\infty}(s) > 1/C$ and $\tilde{w}_{+\infty}(s) \le C(1+|y|)$. Now we recall the result of Guo gived by Theorem 2.1 in [Guo90]) :

The only global solution of

$$w'' - \frac{1}{2} y w' + \frac{w}{\beta+1} - \varepsilon w^{-\beta} = 0, \; y \in \mathbb{R},$$

which is greater than or equal to some positive constant c and which grows at most polynomially as $|y| \to \infty$, is $w \equiv (\varepsilon(\beta+1))^{\frac{1}{\beta+1}}$.

Using this result, it follows that $\tilde{w}_{+\infty} = \kappa$ and we conclude that $\tilde{z}_{+\infty} \equiv \kappa^{-1}$. This concludes the proof of (i) of Lemma 3.2.4.

(ii) We recall from Lemma 3.2.2 that

$$E(z(s)) = \int_{\mathbb{R}} \frac{\partial_y z^2}{z^4} \rho dy - \frac{1}{2(\beta+1)} \int_{\mathbb{R}} \frac{1}{z^2} \rho dy - \frac{1}{\beta-1} \int_{\mathbb{R}} z^{\beta-1} \rho dy. \tag{3.35}$$

If z_j converges to κ^{-1}, using Lesbegue's Theorem and Claim 3.2.5, we obtain

$$E(z_j(s)) = \mathbf{E}(w_j(s)) \to E(\kappa^{-1}) = \mathbf{E}(\kappa).$$

In the case, where z_j converges to 0, we write from Lemma 3.2.2 and Claim 3.2.5

$$E(z_j(s)) = \mathbf{E}(w_j(s)) \le C - \frac{1}{\beta-1} \int w^{-(\beta-1)} \rho dy \tag{3.36}$$

Since the integral in (3.36) tends to infinity as $j \to +\infty$ we have :

$$E(z_j(s)) \to -\infty.$$

This concludes the proof of Lemma 3.2.4 and Proposition 3.2.3.■

To end this part, we recall the result obtained by Proposition 3.2.3, which says that $z_{+\infty}(y) = \lim_{s\to+\infty} z(y,s)$ and $z_{-\infty}(y) = \lim_{s\to-\infty} z(y,s)$ exist and equal 0 or κ^{-1}. Now, we will prove that the case $z_{-\infty} = 0$ can't occur.
Indeed, suppose that $z(y,s) \to 0$ as $s \to -\infty$, by (ii) of Lemma 3.2.4, we have :

$$E(z(s)) \to -\infty \text{ as } s \to -\infty.$$

Therefore, there exists $s_1 < 0$, such that for all $s \le s_1$, we have

$$s < 0 \text{ and } E(z(s)) < E(z(0)).$$

which contradicts the monotonicity of E. Thus, the case $z_{-\infty} = 0$ is ruled out. In the following, we will study the case where $w_{-\infty} = \kappa$.

3.2.2 Part 2 : Linear behavior of w near κ as $s \to -\infty$

In this part, we assume that

$$w \to \kappa \text{ as } s \to -\infty \tag{3.37}$$

in L^2_ρ and uniformly on every compact sets, and we classify the L^2_ρ behavior of $w - \kappa$ as $s \to -\infty$. Let us introduce $v = w - \kappa$. From (3.7), v satisfies the following equation :

$$\forall (y,s) \in \mathbb{R}^2,\ \partial_s v = \mathcal{L}v - f(v), \tag{3.38}$$

where

$$\begin{array}{rcl} \mathcal{L}v & = & \partial_y^2 v - \frac{1}{2} y \partial_y v + v \text{ and} \\ f(v) & = & (v+\kappa)^{-\beta} - \kappa^{-\beta} + \beta \kappa^{-(\beta+1)} v. \end{array} \tag{3.39}$$

Concerning the nonlinear term of equation (3.39), we have :

Lemma 3.2.6. *There exists $s_1 \in \mathbb{R}$ such that for all $s \le s_1$:*

$$0 \le f(v) \le M_1 v^2 \text{ and } f(v) \le M_2 |v|. \tag{3.40}$$

Proof : Using Taylor's formula, we obtain

$$(v+\kappa)^{-\beta} = \kappa^{-\beta} - \frac{\beta}{(\beta+1)} v + \frac{1}{2}\beta(\beta+1)(\theta+\kappa)^{-\beta-2} v^2, \tag{3.41}$$

for some θ between 0 and v. Therefore

$$f(v) = c(\theta,\beta) v^2, \qquad c(\theta,\beta) = \frac{1}{2}\beta(\beta+1)(\theta+\kappa)^{-\beta-2}.$$

If $v > 0$ then $0 \le c(\theta,\beta) \le (\beta/2)(\beta+1)\kappa^{-\beta-2} = \beta/(2\kappa)$.
If $v < 0$, recalling that $\frac{1}{M} \le v + \kappa \le \theta + \kappa$ we obtain $0 \le c(\theta,\beta) \le (\beta/2)(\beta+1)M^{\beta+2}$. Thus (3.40) was established. To show the second inequality in (3.40) we observe that if $v < \kappa$, then it follows from (3.40). If $v > \kappa$ then from (3.39) and the lower bound of v, we have

$$f(v) \le \frac{\beta}{\beta+1} v - \frac{1}{\beta+1}\kappa + M^\beta \le M_2 v,$$

for some constant $M_2 > 0$.■

In the following, we will discuss general properties of the operator $\mathcal{L}$. At first we note that it is self-adjoint on L^2_ρ. Its spectrum is $spec(\mathcal{L}) = \{1 - \frac{m}{2} | m \in \mathbb{N}\}$; it consists of eigenvalues. The eigenfunctions of $\mathcal{L}$ are simple and derived from Hermite polynomials. For $1 - \frac{m}{2}$ corresponds the eigenfunction

$$h_m(y) = \sum_{n=0}^{[\frac{m}{2}]} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}.$$

The polynomial h_m satisfies $\int h_n h_m \rho dy = 2^n n! \delta_{nm}$. Let us introduce $k_m = \frac{h_m}{\|h_m\|^2_{L^2_\rho}}$. Since the eigenfunctions of $\mathcal{L}$ span all the space L^2_ρ, we expand v as follows :

$$v(y,s) = \sum_{m=0}^{2} v_m(s) \cdot h_m(y) + v_-(y,s) \tag{3.42}$$

where v_m is the projection of v on h_m and $v_-(y,s) = P_-(v)$, with P_- is the orthogonal projector on the negative subspace of $\mathcal{L}$.

Now we show that as $s \to -\infty$, either $v_0(s)$, $v_1(s)$ or $v_2(s)$ is predominant with respect to the expansion (3.42) of v in L^2_ρ. We have the following :

Proposition 3.2.7. *(Classification of the behavior of $v(y,s)$ as $s \to -\infty$) As $s \to -\infty$, one of the following situations occurs :*
(i) $|v_1(s)| + |v_2(.,s)| + \|v_-(.,s)\|_{L^2_\rho} = o(v_0(s))$, $\|v(.,s) - C_0 e^s\|_{L^2_\rho} = O(e^{3s/2})$ for some $C_0 \in \mathbb{R}$.

(ii) $|v_0(s)| + |v_2(.,s)| + \|v_-(.,s)\|_{L^2_\rho} = o(v_1(s))$, $\|v(.,s) - C_1 y e^s/2\|_{L^2_\rho} = O(e^{(1-\varepsilon)s})$ for some $C_1 \in \mathbb{R} \setminus 0$ and any $\varepsilon > 0$.

(iii) $|v_0(s)| + |v_1(s)| + \|v_-(.,s)\|_{L^2_\rho} = o(\|v_2(.,s)\|_{L^2_\rho})$, $\|v(.,s) - \frac{\kappa}{4\beta s}(y^2 - 2)\|_{L^2_\rho} = O(\frac{\log(|s|)}{s^2})$.

Proof : See the Appendix.■

3.2.3 Part 3 : Case (i) of Proposition 3.2.7 : $\exists s_0 \in \mathbb{R}$ such that $w(y,s) = \varphi(s - s_0)$

In this part, we prove the following :

Proposition 3.2.8. *(The relevant case (i) of Proposition 3.2.7) Assume that case (i) of Proposition 3.2.7 holds. Then there exists $s_0 \in \mathbb{R}$ such that for all $(y,s) \in \mathbb{R}^2$, $w(y,s) = \varphi(s - s_0)$, where $\varphi(s)$ is introduced in (3.13).*

Proof : First we recall from (i) of Proposition 3.2.7 and the definition of v that

$$\|w(.,s) - \{\kappa + C_0 e^s\}\|_{L^2_\rho} \le C e^{3s/2} \text{ as } s \to -\infty. \tag{3.43}$$

Let us remark that we already have a solution $\hat{\varphi}$ of (3.7) defined in $\mathbb{R} \times (-\infty, s_1]$ for some $s_1 \in \mathbb{R}$ and which satisfies the same expansion as $w(y,s)$ when $s \to -\infty$:

$$\begin{aligned}
&\text{(a) if} \quad C_0 = 0, \text{ just take } \hat{\varphi} \equiv \kappa,\\
&\text{(b) if} \quad C_0 > 0, \text{ take } \hat{\varphi} \equiv \varphi(s - s_0) \text{ where } s_0 = -\log\left(\frac{(\beta+1)}{\kappa} C_0\right),\\
&\text{(c) if} \quad C_0 < 0, \text{ take } \hat{\varphi} \equiv \varphi_0(s - s_0) \text{ where } s_0 = -\log\left(-\frac{(\beta+1)}{\kappa} C_0\right)\\
&\qquad \text{and } \varphi_0(s) = \kappa(1 - e^s)^{\frac{1}{\beta+1}}
\end{aligned} \tag{3.44}$$

is a solution of (3.7) that quenches at $s = 0$ but there exist $C > 0$ such that $\varphi_0 \geq C$ for all $s \leq -1$. If we introduce $V = w - \hat{\varphi}$, then we see from (3.7) that V is defined for all $(y, s) \in \mathbb{R} \times (-\infty, s_1]$ and satisfies

$$\frac{\partial V}{\partial s} = (\mathcal{L} + l(s))V - F(V), \tag{3.45}$$

where $\mathcal{L}$ is given in (3.39)

$$F(V) = |\hat{\varphi} + V|^{-\beta} - \hat{\varphi}^{-\beta} + \beta\hat{\varphi}^{-(\beta+1)}V$$

and

$$\begin{aligned} l(s) &= 0 && \text{if } \hat{\varphi} = \kappa, \\ l(s) &= -\frac{\beta e^s}{(\beta+1)(1+e^s)} && \text{if } \hat{\varphi} = \varphi(s - s_0), \\ l(s) &= \frac{\beta e^s}{(\beta+1)(1-e^s)} && \text{if } \hat{\varphi} = \varphi_0(s - s_0). \end{aligned} \tag{3.46}$$

We note that $|l(s)| \leq Ce^s$ for all $s \leq s_1$ and some $C > 0$ and there exist $M_{i=1,2}$ such that

$$0 \leq F(V) \leq M_1 V^2 \text{ and } 0 \leq F(V) \leq M_2|V|. \tag{3.47}$$

We omit the proof of (3.47) since it is quite similar to the proof of (3.40). Let us introduce $I(s) = \|V(., s)\|_{L^2_\rho}$, multiply (3.45) by $V\rho$ and integrate over $\mathbb{R}$. Using the fact that 1 is the greatest eigenvalue of $\mathcal{L}$ and (3.47), we obtain

$$I'(s) \leq (1 + l(s))I(s) + C\int V^4(y, s)\rho dy.$$

Now, we recall the following from [Vel93] :

Lemma 3.2.9. *(Regularizing effect of the operator $\mathcal{L}$) Assume that $\psi(y, s)$ satisfies*

$$\forall s \in [a, b], \quad \forall y \in \mathbb{R}, \quad \partial_s\psi \leq (\mathcal{L} + \sigma)\psi,\ 0 \leq \psi(y, s),$$

for some $a \leq b$ and $\sigma \in \mathbb{R}$, where

$$\mathcal{L}\psi = \partial_y^2\psi - \frac{1}{2}y \cdot \partial_y\psi + \psi = \frac{1}{\rho}\text{div}(\rho\partial_y\psi) + \psi. \tag{3.48}$$

Then for any $r > 1$, there exists $C^* = C^*(r, \sigma) > 0$ and $s^*(r)$, such that

$$\forall s \in [a + s^*, b], \quad \left(\int_{\mathbb{R}} |\psi(y, s)|^r \rho(y)dy\right)^{1/r} \leq C^*\|\psi(., s - s^*)\|_{L^2_\rho}. \tag{3.49}$$

Proof of Lemma : See Lemma 2.3 in [HV93]■

Using the lemma above, we obtain the existence of $C^* > 0$ and $s^* > 0$, such that $\|V(., s)^2\|_{L^2_\rho} \leq C^*\|V(., s - s^*)\|^2_{L^2_\rho}$. Then we obtain for some $s_2 \leq s_1$:

$$\forall s \leq s_2 \quad I'(s) \leq \frac{5}{4}I(s) + CI(s - s^*)^2. \tag{3.50}$$

Since $I(s) \leq Ce^{3s/2}$ from (3.43), the following lemma from [NZ08] allows us to conclude.

Lemma 3.2.10. *(Lemma 3.6 of [NZ08]) Consider $I(s)$ a positive C^1 function such that (3.50) is satisfied and $0 \le I(s) \le Ce^{3s/2}$ for all $s \le s_2$, for some s_2. Then, for some $s_3 \le s_2$, we have $I(s) = 0$ for all $s \le s_3$.*

Using Lemma 3.2.10, we obtain $V \equiv 0$ on $\mathbb{R} \times (-\infty, s_3]$. Consequently, we have

$$\forall (y,s) \in \mathbb{R} \times (-\infty, s_3], \quad w(y,s) = \hat{\varphi}(s). \tag{3.51}$$

From the uniqueness of the Cauchy problem for equation (3.7) and since w is defined for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, $\hat{\varphi}$ is defined for all $(y,s) \in \mathbb{R}\times\mathbb{R}$ and (3.51) holds for all $(y,s) \in \mathbb{R}\times\mathbb{R}$. Therefore, case (c) in (3.46) cannot hold and for all $(y,s) \in \mathbb{R}^2$, $w(y,s) = \kappa$ or $w(y,s) = \varphi(s - s_0)$. This concludes the proof of Proposition 3.2.8 and finishes Part 3.

3.2.4 Part 4 : Irrelevance of the case (iii) of Proposition 3.2.7

We consider case (iii) of Proposition 3.2.7. In this part, we will proceed like in Step 4 in [NZ08]. The following proposition allows us to reach a contradiction.

Proposition 3.2.11. *Assume that case (iii) of Proposition 3.2.7 holds. Then, there exists $\varepsilon_0 > 0$ such that*

$$\begin{aligned} &\lim_{s\to-\infty} \sup_{|y|\le\varepsilon_0\sqrt{-s}} \left| w(y,s) - G\left(\frac{y}{\sqrt{-s}}\right)\right| = 0, \\ &\textit{where } G(\xi) = \left((\beta+1) - \tfrac{(\beta+1)^2}{4\beta}\xi^2\right)^{\frac{1}{\beta+1}}. \end{aligned} \tag{3.52}$$

Indeed, let us first use Proposition 3.2.11 to find a contradiction ruling out case (iii) of Proposition 3.2.7, and then prove Proposition 3.2.11.
We fundamentally rely on the following Lemma :

Lemma 3.2.12. *(Lemma 2.11 from [MZ98b]) Assume that $\psi(\xi,\tau)$ satisfies for all $|\xi| \le 4B_1$ and $\tau \in [0,\tau_*]$:*

$$\begin{cases} \partial_\tau \psi & \le \partial^2_{yy}\psi + \lambda\psi + \mu, \\ \psi(\xi,0) & \le \psi_0,\ \psi(\xi,\tau) \le B_2, \end{cases}$$

where $\tau_* \le 1$. Then, for all $|\xi| \le B_1$ and $\tau \in [0,\tau_*]$,

$$\psi(\xi,\tau) \le e^{\lambda\tau}(\psi_0 + \mu + CB_2 e^{-B_1^2/4}).$$

Let us define u_{s_0} by

$$u_{s_0}(\xi,\tau) = (1-\tau)^{\frac{1}{\beta+1}} w(y,s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}}{\sqrt{1-\tau}} \text{ and } s = s_0 - \log(1-\tau). \tag{3.53}$$

We note that u_{s_0} is defined for all $\tau \in [0,1)$ and $\xi \in \mathbb{R}$, and that u_{s_0} satisfies equation (3.1). From Lemma 3.2.1, we have

$$\forall \tau \in [0,1),\ |u_{s_0}(.,\tau)| \ge M(1-\tau)^{\frac{1}{\beta+1}}. \tag{3.54}$$

The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}, s_0)$. Using Proposition 3.2.11, we get :

$$\sup_{|\xi|<4|s_0|^{1/4}} \left| u_{s_0}(\xi, 0) - G\left(\frac{\varepsilon_0}{2}\right) \right| \equiv g(s_0) \to 0 \text{ as } s_0 \to -\infty. \tag{3.55}$$

If we define v, the solution of :

$$v' = -v^{-\beta} \text{ and } v(0) = G\left(\frac{\varepsilon_0}{2}\right),$$

then

$$v(\tau) = \kappa\left(1 - \frac{(\beta+1)\varepsilon_0^2}{16\beta} - \tau\right)^{\frac{1}{\beta+1}}, \tag{3.56}$$

which quenches at time $1 - \frac{(\beta+1)\varepsilon_0^2}{16\beta} < 1$. Therefore, there exists $\tau_0 = \tau_0(\varepsilon_0) < 1$, such that

$$v(\tau_0) = \frac{M}{3}(1-\tau_0)^{\frac{1}{\beta+1}}. \tag{3.57}$$

Now, if we consider the function

$$\psi = |u_{s_0} - v|, \tag{3.58}$$

then the following claim allows us to conclude :

Claim 3.2.13. *For $|s_0|$ large enough, $\tau \in [0, \tau_0]$ and $|\xi| \le 4|s_0|^{1/4}$, we have :*
(i) $\partial_\tau \psi \le \partial_\xi^2 \psi + C(\varepsilon_0)\psi$,
(ii) $\psi(\xi, 0) \le g(s_0)$,
(iii) $\psi(\xi, \tau) \le C(\varepsilon_0)|s_0|^{1/2}$.

Indeed, using Lemma 3.2.12 with $B_1 = |s_0|^{1/4}$, $B_2 = C(\varepsilon_0)|s_0|^{1/2}$, $\tau_* = \tau_0$, $\psi_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$, we get for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi|\le|s_0|^{1/4}} \psi(\xi, \tau) \le C(\varepsilon_0)\left(g(s_0) + |s_0|^{1/2} e^{-\frac{|s_0|^{1/2}}{4}}\right) \to 0 \text{ as } s_0 \to -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get : $\psi(0, \tau_0) \le \frac{M}{3}(1-\tau_0)^{1/(\beta+1)}$ and by (3.57)

$$|u_{s_0}(0, \tau_0)| \le v(\tau_0) + \psi(0, \tau_0) \le \frac{2}{3}M(1-\tau_0)^{1/(\beta+1)},$$

which is in contradiction with (3.54). To conclude, it remains to prove Claim 3.2.13.

Proof of Claim 3.2.13 : (i) If we note by $\Psi = u_{s_0} - v$, then we get

$$\begin{aligned} \partial_s \Psi &= \partial_s^2 \Psi - \left(u_{s_0}^{-\beta} - v^{-\beta}\right), \\ &= \partial_y^2 \Psi + \beta \Psi \theta^{-\beta+1}, \text{ for some } \theta \text{ between } u_{s_0} \text{ and } v. \end{aligned}$$

Using (3.54), (3.56) and (3.57) , we have for all $\tau \in [0, \tau_0]$ and $\xi \in \mathbb{R}$

$$\frac{1}{\theta} \le \max\left(\frac{1}{u_0(\xi, \tau)}, \frac{1}{v(\tau)}\right) \le C(\varepsilon_0), \text{ for some positive } \theta.$$

Since $\psi = |\Psi|$, using Kato's inequality, we conclude the proof of (i).
(ii) It is directly obtained from (3.55).
(iii) Using the definition (3.58) of ψ, (3.53) and (3.56), we write for all $\tau \in [0, \tau_0]$ and $|\xi| \leq 4|s_0|^{1/4}$,

$$\psi(\xi, \tau) \leq u_{s_0}(\xi, \tau) + v(\tau) \leq w(y, s) + \kappa, \text{ where } (y, s) \text{ are defined in (3.53).} \tag{3.59}$$

Since we have from Lemma 3.2.1, (3.37) and (3.53) for $|s_0|$ large enough

$$|w(y, s)| \leq |w(0, s)| + M_0|y| \leq C(1 + |y|)$$

and

$$|y| \leq \frac{|\xi + \frac{\varepsilon_0}{2}\sqrt{-s_0}|}{\sqrt{1-\tau}} \leq \frac{4|s_0|^{1/4} + \frac{\varepsilon_0}{2}\sqrt{-s_0}}{\sqrt{1-\tau_0}} \leq C(\varepsilon_0)|s_0|^{1/2},$$

the bound on ψ follows from (3.59). This concludes the proof of claim 3.2.13. ■

It remains to prove Propossition 3.2.11 to conclude Part 4.
Proof of Proposition 3.2.11 : Consider some arbitrary $\varepsilon_0 \in (0, R^*)$, where

$$R^* = \sqrt{\frac{4\beta}{\beta + 1}}. \tag{3.60}$$

The parameter ε_0 will be fixed later in the proof small enough. If we note

$$f(y, s) = G\left(\frac{y}{\sqrt{-s}}\right), \tag{3.61}$$

and

$$F(y, s) = f(y, s) - \frac{\kappa}{2\beta s}, \tag{3.62}$$

then f satisfies

$$-\frac{y}{2} \cdot \partial_y f + \frac{1}{(\beta + 1)} f - |f|^{-\beta} = 0.$$

and we see from (iii) of Proposition 3.2.7 that

$$\| \left(F(., s) - w(., s)\right)(1 - \chi_{\varepsilon_0})\|_{L^2_\rho} = O\left(\frac{\log|s|}{s^2}\right) \text{ as } s \to -\infty, \tag{3.63}$$

where

$$\chi_{\varepsilon_0}(y, s) = 1 \text{ if } \frac{|y|}{\sqrt{|s|}} \geq 3\varepsilon_0 \text{ and zero otherwise.} \tag{3.64}$$

The formal idea of this proof is that F is an approximate solution (as $s \to -\infty$) of equation (3.7) as w. By (3.63), w and F are very close in the region $|y| \sim 1$. Our task is to prove that they remain close in the larger region $|y| \leq \varepsilon_0\sqrt{-s}$, for some ε_0 chosen later. Let us consider a cut-off function

$$\gamma(y, s) = \gamma_0\left(\frac{y}{\sqrt{-s}}\right), \tag{3.65}$$

where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\gamma_0(\xi) = 1$ if $|\xi| \le 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \ge 4\varepsilon_0$. We introduce

$$\nu = w - F \text{ and } Z = \gamma|\nu|. \tag{3.66}$$

Our proof is the same as Velázquez [Vel92] and Nouaili and Zaag [NZ08]. As in [NZ08], we need to multiply by the cut-off, since our profile $F(y,s)$ defined by (3.62) is singular on the parabola $y = R^*\sqrt{-s}$. The cut-off function will generate an extra term, difficult to handle. Let us present the major steps of the proof in the following. The proof of the presented Lemmas will be given at the end of this step.

Lemma 3.2.14. (Estimates in modified L^2_ρ spaces) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \le s_*$ and $y \in \mathbb{R}$:*

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y\cdot\partial_y Z - (1+\sigma)Z \le C\left(Z^2 + \frac{(y^2+1)}{s^2} + (1+\sqrt{-s})\chi_{\varepsilon_0}\right) - 2\text{div}\left(|\nu|\cdot\partial_y\gamma\right), \tag{3.67}$$

where $s_* \in \mathbb{R}$, $\sigma = 1/100$ and χ_{ε_0} is defined in (3.64). Moreover,

$$N^2_{2\varepsilon_0\sqrt{|s|}}(Z(s)) = o(1) \text{ as } s \to -\infty, \tag{3.68}$$

where the norm $N^q_r(\psi)$ is defined, for all $r > 0$ and $1 \le q < \infty$, by

$$N^q_r(\psi) = \sup_{|\xi|\le r}\left(\int |\psi(y)|^q \exp\left(-\frac{(y-\xi)^2}{4}\right)dy\right)^{1/q}. \tag{3.69}$$

Using the regularizing effect of the operator $\mathcal{L}$, we derive the following pointwise estimate, which allows us to conclude the proof of Proposition 3.2.11 :

Lemma 3.2.15. (An upper bound for $Z(y,s)$ in $\{|y| \le \varepsilon_0\sqrt{-s}\}$) *We have :*

$$\sup_{|y|\le\varepsilon_0\sqrt{-s}} Z(y,s) = o(1) \text{ as } s \to -\infty.$$

Thus, Proposition 3.2.11 follows from Lemma 3.2.15 by (3.66), (3.62) and (3.61). It remains to prove Lemmas 3.2.14 and 3.2.15.

Proof of Lemma 3.2.14 : The proof of (3.67) is straightforward and a bit technical. We leave it to Appendix 3.3.2. Let us then prove (3.68). We take $s_0 < s^*$ and $s_0 \le s < s^*$ such that $e^{\frac{s-s_0}{2}} \le \sqrt{-s}$. We use the variation of constant formula in (3.67) to write

$$\begin{aligned} Z(y,s) \le\ & S_\sigma(s-s_0)Z(.,s_0) \\ & + \int_{s_0}^{s} S_\sigma(s-\tau)\left(C\left\{Z^2 + \frac{(y^2+1)}{\tau^2} + (1+\sqrt{-\tau})\chi_{\varepsilon_0}\right\} - 2\text{div}\left(|\nu|\partial_y\gamma\right)\right)d\tau, \end{aligned}$$

where S_σ is the semigroup associated to the operator $\mathcal{L}_\sigma\phi = \partial_y^2\phi - \frac{1}{2}y\partial_y\phi + (1+\sigma)\phi$, defined on $L^2_\rho(\mathbb{R})$. The kernel of the semigroup $S_\sigma(\tau)$ is

$$S_\sigma(\tau, y, z) = \frac{e^{(1+\sigma)\tau}}{(4\pi(1-e^{-\tau}))^{1/2}} \exp\left[-\frac{|ye^{-\tau/2} - z|^2}{4(1-e^{-\tau})}\right]. \tag{3.70}$$

Setting

$$r \equiv r(s, s_0) = 2\varepsilon_0 e^{\frac{s-s_0}{2}} = R_1 e^{\frac{s-s_0}{2}} \tag{3.71}$$

and taking the N_r^2−norm we obtain

$$\begin{aligned} N_r^2(Z(.,s)) &\le N_r^2(S_\sigma(s-s_0)Z(.,s_0)) + C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)Z(.,\tau)^2)d\tau \\ &\quad + C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)\left(\frac{(y^2+1)}{\tau^2}\right))d\tau \\ + C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)(1+\sqrt{-\tau})\chi_{\varepsilon_0}(y,\tau))d\tau &+ C\int_{s_0}^{s} N_r^2(S_\sigma(s-\tau)(\mathrm{div}(|\nu|\partial_y\gamma))d\tau \\ &\equiv J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

In comparison with [Vel92], we have a new term J_5 coming from the cut-off terms. Therefore, we just recall in the following claim the estimates on $J_1 ... J_4$ from [Vel92], and treat J_2 and J_5, which are new ingredients in our proof :

Claim 3.2.16. *We obtain as $s \to -\infty$*

$$\begin{aligned} |J_1| &\le Ce^{(1+\sigma)(s-s_0)}\frac{\log|s_0|}{|s_0^2|}, \\ |J_2| &\le C\int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(1+\sigma)(s-\tau-R_0)}}{(1-e^{s-\tau-R_0})^{1/20}}\left(N_r^2(Z(.,\tau)^2)\right)d\tau + C\frac{e^{(s-s_0)(1+\sigma)}}{s_0^2}, \\ & \text{with } R_0 = 4\varepsilon_0, \\ |J_3| &\le C\frac{e^{(s-s_0)(1+\sigma)}}{s_0^2}\left(1+(s-s_0)\right)\sqrt{-s_0}, \\ |J_4| &\le Ce^{(s-s_0)(1+\sigma)}e^{\alpha s}, \text{ where } \alpha > 0, \\ |J_5| &\le Ce^{(s-s_0)(1+\sigma)}e^{\beta s}, \text{ where } \beta > 0. \end{aligned}$$

Proof : See page 1578 in [Vel92] for J_1, J_3 and J_4.
To obtain the bound on J_2, we need the following inequality, which will be proved in Appendix 3.3.2

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2}y\partial_y Z \le C\left(Z + \frac{(y^2+1)}{s^2} + (1+\sqrt{-s})\chi_{\varepsilon_0}\right) - 2\mathrm{div}\left(|\nu|\cdot\partial_y\gamma\right), \tag{3.72}$$

then, we proceed exactly as in page 1579-1580 in [Vel92] and we obtain the wanted estimation on J_2.

Now, we treat J_5. We have from (3.70) :

$$\begin{aligned} &S_\sigma(s-\tau)\left(-\mathrm{div}(|\nu\partial_y\gamma)\right), \\ &= -\frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\mathrm{div}(|\nu|\partial_y\gamma)d\lambda, \\ &= \frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}} -\frac{(ye^{-(s-\tau)/2}-\lambda)}{2(1-e^{-(s-\tau)})}\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)|\nu|\partial_y\gamma d\lambda. \end{aligned} \tag{3.73}$$

We have from Lemma 3.2.1 and (3.37)

$$|w(y,s)| \le |w(0,s)| + M_0|y| \le C(1+|y|).$$

Since F is bounded for $\frac{|y|}{\sqrt{-\tau}} \le R^*/2$, where R^* is defined by (3.60) and $supp(\partial_y\gamma) \subset (-4\varepsilon_0\sqrt{-\tau}, -3\varepsilon_0\sqrt{-\tau}) \cup (3\varepsilon_0\sqrt{-\tau}, 4\varepsilon_0\sqrt{-\tau})$, we have

$$\begin{aligned} ||\nu|\partial_y\gamma\ | &\le C|\nu|\mathbb{I}_{\{3\varepsilon_0\le\frac{|y|}{\sqrt{-\tau}}\le 4\varepsilon_0\}}, \\ &\le C(1+\sqrt{-\tau})(\mathbb{I}_{3\varepsilon_0\le\frac{|y|}{\sqrt{-\tau}}\le 4\varepsilon_0}) \le C(1+\sqrt{-\tau})\chi_{\varepsilon_0}. \end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain :

$$|S_\sigma(s-\tau)\left(-\mathrm{div}(|\nu|\partial_y\gamma)\right)| \le \frac{Ce^{(s-\tau)(1+\sigma)}(1+\sqrt{-\tau})}{(1-e^{s-\tau})^{3/2}}\mathcal{I}_1\mathcal{I}_2,$$

where,

$$\begin{aligned} \mathcal{I}_1 &= \left(\int_{\mathbb{R}}(ye^{-(s-\tau)/2}-\lambda)^2\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)d\lambda\right)^{1/2}, \\ \mathcal{I}_2 &= \left(\int_{\mathbb{R}}\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\chi_{\varepsilon_0}d\lambda\right)^{1/2}. \end{aligned}$$

Doing a change of variables, we obtain $\mathcal{I}_1 = C(1-e^{-(s-\tau)})^{3/4}$. Furthermore, we have :

$$\mathcal{I}_2^2 \le \mathcal{I}_3\left(\int_{\mathbb{R}}\chi_{\varepsilon_0}e^{-\frac{\lambda^2}{4}}d\lambda\right)^{1/2},$$

where,

$$\mathcal{I}_3 = \left(\int\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{2(1-e^{-(s-\tau)})}+\frac{\lambda^2}{4}\right)d\lambda\right)^{1/2}.$$

We introduce $\theta = ye^{-(s-\tau)/2}$, by completing squares, we readily check that :

$$\frac{\lambda^2}{4} - \frac{(\theta-\lambda)^2}{2(1-e^{-(s-\tau)})} = -\frac{(1+e^{-(s-\tau)})}{4(1-e^{-(s-\tau)})}(\lambda - \frac{2\theta}{(1+e^{-(s-\tau)})})^2 + \frac{\theta^2}{2(1+e^{-(s-\tau)})},$$

then we obtain :

$$\mathcal{I}_3^2 = C\left(\frac{(1-e^{-(s-\tau)})}{(1+e^{-(s-\tau)})}\right)^{1/2}\exp\left(\frac{\theta^2}{2(1-e^{-(s-\tau)})}\right).$$

Therefore,

$$\begin{aligned}\left|N_r^2(S_\sigma(s-\tau)\mathrm{div}(|\nu|\partial_y\gamma))\right| &\leq C\frac{e^{(s-\tau)(1+\sigma)}(1+\sqrt{-\tau})}{(1+e^{-(s-\tau)})^{1/8}(1-e^{-(s-\tau)})^{5/8}}\|\chi_{\varepsilon_0}\|_{L^2_\rho}^{1/2}\mathcal{I}_4,\\ \text{where } \mathcal{I}_4 &= N_r^2\left(\exp(\frac{y^2e^{-(s-\tau)}}{8((1-e^{-(s-\tau)}))})\right).\end{aligned}$$

Let us compute $\mathcal{I}_4$. Using the fact that

$$-\frac{(y-\mu)^2}{4}+\frac{y^2e^{-(s-\tau)}}{4(1-e^{-(s-\tau)})}=$$
$$\frac{1}{4}\left(-\left(y(1+e^{-(s-\tau)})^{-1/2}-\mu(1+e^{-(s-\tau)})^{1/2}\right)^2+\mu^2e^{-(s-\tau)}\right),$$

and doing a change of variables, we obtain :

$$\begin{aligned}&\int_{\mathbb{R}}\exp\left(-\frac{(y-\mu)^2}{4}+\frac{y^2e^{-(s-\tau)}}{4(1-e^{(s-\tau)})}\right)dy\\ &\leq C\exp\left(\frac{\mu^2e^{-(s-\tau)}}{4}\right)\int_{\mathbb{R}}\exp\left(-\frac{1}{4}\left(y(1+e^{-(s-\tau)})^{-1/2}-\mu(1+e^{-(s-\tau)})^{1/2}\right)^2\right)dy.\end{aligned}$$

Hence $\mathcal{I}_4\leq C(1+e^{-(s-\tau)})^{1/8}$ and

$$N_r^2(S_\sigma(s-\tau))\mathrm{div}(|\nu|\partial_y\gamma))\leq C\frac{e^{(s-\tau)(1+\sigma)}(1+\sqrt{-\tau})}{(1-e^{-(s-\tau)})^{5/8}}\left(\int_{|\lambda|\geq R_1\sqrt{-\tau}}e^{-\frac{\lambda^2}{4}}d\lambda\right).$$

This gives

$$|J_5| = \int_{s_0}^{s}N_r^2\left(S_\sigma(s-\tau)(\mathrm{div}(|\nu|\partial_y\gamma)\right)d\tau\leq C(\eta)e^{(s-s_0)(1+\sigma)}e^{\alpha s_0},$$

where $\alpha>0$. This concludes the proof of the claim 3.2.16.
This concludes the proof of Claim 3.2.16.■
Summing up $J_{i=1..5}$, from claim 3.2.16 we obtain

$$\begin{aligned}&N_r^2(Z(.,s))\leq\\ &e^{(s-s_0)(1+\sigma)}C\frac{\log|s_0|}{s_0^2}+C\int_{s_0}^{s_0+((s-R_0)-s_0)_+}\frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}}\left(N_r^2(Z(.,\tau))\right)^2d\tau.\end{aligned}$$

Now, we recall the following from [Vel92] :

Lemma 3.2.17. *Let ε, C, R, σ and α be positives constants, $0<\alpha<1$ and assume that $H(s)$ is a family of continuous functions satisfying :*

$$H(s)\leq\varepsilon e^{s(1+\sigma)}+C\int_0^{(s-R)_+}\frac{e^{(s-\tau)(1+\sigma)}H(\tau)^2}{(1-e^{(s-\tau-R)})^\alpha}d\tau \text{ for } s>0.$$

Then there exists $\xi=\xi(R,C,\alpha)$ such that for any $\varepsilon\in(0,\varepsilon_1)$ and any s for which $\varepsilon e^{s(1+\sigma)}\leq\xi$, we have

$$H(s)\leq 2\varepsilon e^{s(1+\sigma)}.$$

Proof : See the proof of Lemma 2.2 from [Vel92]. Note that the proof of [Vel92] is done in the case $\sigma = 0$, but it can be adapted to some $\sigma > 0$ with no difficulty.■

We conclude that $N^2_{r(\tau,s_0)}(Z(.,s)) \le Ce^{(s-s_0)(1+\sigma)}\frac{\log|s_0|}{s_0^2}$ as $s \to -\infty$. If we fix $s = -e^{(s-s_0)}$, then we obtain $s \sim s_0$, $\log|s| \sim \log|s_0|$ and $N^2_{R_1\sqrt{-s}}(Z(.,s)) \le Cs^{1+\sigma}\frac{\log|s_0|}{s_0^2} \le C\frac{\log|s|}{s^{1-\sigma}} \to 0$ as $s \to -\infty$. Since $\sigma = \dfrac{1}{100}$, we get $N^2_{R_1\sqrt{-s}}(Z(.,s)) = o(1)$, as $s \to -\infty$.
This concludes the proof of Lemma 3.2.14.■

Now we give the proof of Lemma 3.2.15.
Proof of Lemma 3.2.15.
We aim at bounding $Z(y,s)$ for $|y| \le R_2\sqrt{-s}$ in terms of $N_{R_1\sqrt{-s'}}(Z(s'))$, where $R_2 = \varepsilon_0$ and $R_1 = 2\varepsilon_0$, for some $s' < s$. Starting from equation (3.72), we do as in [Vel92] :

$$\begin{aligned} Z(.,s) &\le \left\{e^{CR_0}S(R_0)Z(.,s-R_0)\right\} \\ &\quad + \left\{C\int_{s-R_0}^{s} e^{C(s-\tau)}S(s-\tau)\left(\frac{(y^2+1)}{\tau^2} + (1+\sqrt{-\tau})\chi_{\varepsilon_0}\right)d\tau\right\} \\ &\quad - \left\{2\int_{s-R_0}^{s} e^{C(s-\tau)}S(s-\tau)\left(\operatorname{div}(|\nu|\partial_y\gamma)\right)d\tau\right\} \\ &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \text{ where } R_0 = 4\varepsilon_0, \end{aligned}$$

where S is the semigroup associated to the operator $\mathcal{L}$ defined in (3.48). The terms $\mathcal{M}_1$ and $\mathcal{M}_2$ are estimated in the following :

Claim 3.2.18. *(Velázquez) There exists s_0, such that for all $s \le s_0$*

$$\begin{aligned} \sup_{|y|\le R_2\sqrt{-s}} |\mathcal{M}_1| &= \sup_{|y|\le R_2\sqrt{-s}} |e^{CR_0}S(R_0)Z(.,s-R_0)| = o(1) \ \text{as } s \to -\infty, \\ \sup_{|y|\le R_2\sqrt{-s}} |\mathcal{M}_2| &= \sup_{|y|\le R_2\sqrt{-s}} \int_{s-R_0}^{s}\left(\tfrac{|y|^2+1}{s^2} + (1+\sqrt{-\tau})\chi_{\varepsilon_0}\right) \le \tfrac{C}{|s|}. \end{aligned} \quad (3.74)$$

Proof : See page 1581 from [Vel92] and Lemma 6.5 in [HV93] in a similar case.■
It remains to estimate $\mathcal{M}_3$. Proceeding as in page (3.73) and using the fact that $\nu(y,s) \le C(1+\sqrt{-s})$ for all $|y| \le C\sqrt{-s}$ (obtained from (i) of Lemma 3.2.1) we write

$$\begin{aligned} &|S(s-\tau)\,(- \div (|\nu|\partial_y\gamma))| \\ &= \left|\frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\operatorname{div}(|\nu|\partial_y\gamma)d\lambda\right|, \\ &= \left|\frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}}\int_{\mathbb{R}} -\frac{(ye^{-(s-\tau)/2}-\lambda)}{2(1-e^{-(s-\tau)})}\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)(|\nu|\partial_y\gamma d\lambda\right|, \\ &\le \frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{3/2}}\int_{\mathbb{R}} |ye^{-(s-\tau)/2}-\lambda|\exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\chi_{\varepsilon_0}d\lambda, \\ &\le \frac{Ce^{s-\tau}\sqrt{-\tau}(1+\sqrt{-\tau})}{(1-e^{s-\tau})^{3/2}}\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\chi_{\varepsilon_0}d\lambda. \end{aligned}$$

We make the change of variables $z = (1-e^{-(s-\tau)})^{-1/2}(\lambda - e^{-(\tau-s)/2}y)$ and we obtain

$$\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2}-\lambda)^2}{4(1-e^{-(s-\tau)})}\right)\chi_{\varepsilon_0}d\lambda \le (1-e^{s-\tau})^{1/2}\int_{\Sigma} e^{-z^2/4}dz,$$

where,

$$\Sigma = \left\{ z \in \mathbb{R} : \left| z + e^{-(\tau-s)/2}(1-e^{s-\tau})^{-1/2} y \right| \geq 3\varepsilon_0 (1-e^{s-\tau})^{-1/2}\sqrt{-\tau} \right\}.$$

Since $|ye^{-(\tau-s)/2}| \leq \varepsilon_0\sqrt{-s}$, we readily see that $\Sigma \subset \left\{ z \in \mathbb{R} : |z| \geq \varepsilon_0\sqrt{-s} \right\}$. Then we conclude that

$$|S(s-\tau)\left(-\mathrm{div}(|\nu|\partial_y\gamma)\right)| \leq \frac{Ce^{s-\tau}}{(1-e^{s-\tau})} e^{\beta s}, \text{ where } \beta > 0,$$

and we obtain

$$\sup_{|y| \leq R_2\sqrt{-s}} |\mathcal{M}_3| = o\left(\frac{1}{|s|}\right) \text{ as } s \to -\infty.$$

Putting together $\mathcal{M}_{i=1..3}$, the proof of lemma 3.2.15 is complete. This concludes also the proof of Proposition 3.2.11 and rules out case (iii) of Proposition 3.2.7.■

3.2.5 Part 5 : Irrelevance of the case (ii) of Proposition 3.2.7

To conclude the proof of Theorem 4, we consider case (ii) of Proposition 3.2.7. We claim that the following proposition allows us to reach a contradiction in this case.

Proposition 3.2.19.

$$\lim_{s\to-\infty} \sup_{|y| \leq \varepsilon_0 e^{-s/2}} \left| w(y,s) - G(ye^{-s/2}) \right| = 0, \text{ where } G(\xi) = \kappa\left(1 - C_1\kappa^\beta\xi\right)^{\frac{1}{\beta+1}}. \quad (3.75)$$

As in the previous part, first, we will find a contradiction ruling out case (ii) of Proposition 3.2.7 and then prove Proposition 3.2.19.
Let us define u_{s_0} by

$$u_{s_0}(\xi,\tau) = (1-\tau)^{\frac{1}{\beta+1}} w(y,s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2}e^{-s_0/2}}{\sqrt{1-\tau}} \text{ and } s = s_0 - \log(1-\tau). \quad (3.76)$$

We note that u_{s_0} is defined for all $\tau \in [0,1)$ and $\xi \in \mathbb{R}$, and that u_{s_0} satisfies equation (3.1). From Lemma 3.2.1, we have

$$\forall \tau \in [0,1),\ |u_{s_0}(.,\tau)| \geq M(1-\tau)^{\frac{1}{\beta+1}}. \quad (3.77)$$

The initial condition at time $\tau = 0$ is $u_{s_0}(\xi,0) = w(\xi + \frac{\varepsilon_0}{2}e^{-s_0/2}, s_0)$. Using Proposition 3.2.11, we get :

$$\sup_{|\xi|<4e^{-s_0/4}} \left| u_{s_0}(\xi,0) - G\left(\frac{\varepsilon_0}{2}\right) \right| \equiv g(s_0) \to 0 \text{ as } s_0 \to -\infty. \quad (3.78)$$

If we define v, the solution of :

$$v' = -v^{-\beta} \text{ and } v(0) = G\left(\frac{\varepsilon_0}{2}\right),$$

then

$$v(\tau) = \kappa \left(1 - C_1\kappa^\beta \frac{\varepsilon_0}{2} - \tau\right)^{\frac{1}{\beta+1}}, \tag{3.79}$$

which quenches at time $1 - C_1\kappa^\beta \frac{\varepsilon_0}{2} < 1$. Therefore, there exists $\tau_0 = \tau_0(\varepsilon_0) < 1$, such that

$$v(\tau_0) = \frac{M}{3}(1-\tau_0)^{\frac{1}{\beta+1}}. \tag{3.80}$$

Now, if we consider the function

$$\psi = |u_{s_0} - v|, \tag{3.81}$$

then the following claim allows us to conclude (we omit the proof since it is the same as the proof of Lemma 3.2.13) :

Claim 3.2.20. *For $|s_0|$ large enough and for all $\tau \in [0, \tau_0]$ and $|\xi| \le 4e^{-s_0/4}$, we have :*
(i) $\partial_\tau \psi \le \partial_\xi^2 \psi + C(\varepsilon_0)\psi$,
(ii) $\psi(\xi, 0) \le g(s_0)$,
(iii) $\psi(\xi,\tau) \le C(\varepsilon_0)e^{-s_0/2}$.

Indeed, using Lemma 3.2.12 with $B_1 = e^{-s_0/4}$, $B_2 = C(\varepsilon_0)e^{-s_0/2}$, $\tau_* = \tau_0$, $\psi_0 = g(s_0)$, $\lambda = C(\varepsilon_0)$ and $\mu = 0$, we get for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi| \le e^{-s_0/4}} \psi(\xi,\tau) \le C(\varepsilon_0)\left(g(s_0) + e^{-s_0/2}e^{-\frac{e^{-s_0/2}}{4}}\right) \to 0 \text{ as } s_0 \to -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get : $\psi(0,\tau_0) \le \dfrac{M}{3}(1-\tau_0)^{1/(\beta+1)}$ and by (3.57)

$$|u_{s_0}(0,\tau_0)| \le v(\tau_0) + \psi(0,\tau_0) \le \frac{2}{3}M(1-\tau_0)^{1/(\beta+1)},$$

which is in contradiction with (3.54).

Proof of Proposition 3.2.19 : The proof is very similar to that of Proposition 3.2.11. If we note $f(y,s) = G(ye^{s/2})$, then f satisfies

$$-\partial_s f - \frac{1}{2}y \cdot \partial_y f + \frac{f}{\beta+1} - f^{-\beta} = 0. \tag{3.82}$$

Consider an arbitrary $\varepsilon_0 \in (0, \frac{R^*}{10})$, where $R^* = \dfrac{\kappa^p}{C_1}$. ε_0 will be fixed small enough later. Let us consider a cut-off function $\gamma(y,s) = \gamma_0(ye^{s/2})$, where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that $\gamma_0(\xi) = 1$ if $|\xi| \le 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \ge 4\varepsilon_0$. We note $\nu = w - f$ and $Z = \gamma|\nu|$. From (ii) of Proposition 3.2.7, we have

$$\|Z\|_{L^2_\rho} \le Ce^{s(1-\varepsilon)} \text{ as } s \to -\infty, \text{ for some } \varepsilon > 0. \tag{3.83}$$

As in the previous part, we divide our proof in two parts given in the following lemmas.

Lemma 3.2.21. (Estimates in the modified L^2_ρ spaces) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \le s_*$ and $y \in \mathbb{R}$,*

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1+\sigma) Z \le C(Z^2 + e^s + (1+e^{-s/2})\chi_{\varepsilon_0}) - 2\mathrm{div}(|\nu|\partial_y \gamma), \quad (3.84)$$

where $s_* \in \mathbb{R}$, $\sigma = \frac{1}{100}$ and

$$\chi_{\varepsilon_0}(y,s) = 1 \text{ if } |y|e^{s/2} \ge 3\varepsilon_0 \text{ and zero otherwise.} \quad (3.85)$$

Moreover, we have

$$N^2_{2\varepsilon_0 e^{-s/2}}(Z(s)) = o(1) \text{ as } s \to -\infty. \quad (3.86)$$

As in Part 4, the following lemma allows us to conclude the proof of Proposition 3.2.19 :

Lemma 3.2.22. (An upper bound for $Z(y,s)$ in $|y| \le \varepsilon_0 e^{-s/2}$) *We have :*

$$\sup_{|y| \le \varepsilon_0 e^{-s/2}} Z(y,s) = o(1) \text{ as } s \to -\infty. \quad (3.87)$$

Remains to prove Lemmas 3.2.21 and 3.2.22 to conclude the proof of Proposition 3.2.19. Here, we only sketch the proof of Lemma 3.2.21, since it is completely similar to Part 4. We don't give the proof of Lemma 3.2.22. We refer the reader to Part 4 and Proposition 2.4 from Velázquez [Vel92] for similar situations.

Proof of Lemma 3.2.21 : As in the previous step, we leave the proof of (3.84) to Appendix 3.3.2.

Let us now apply the variation of constants formula and take the norm $N^2_{r(s,s_0)}$, where $r(s,s_0)$ is as in (3.71). Assume that $s_0 < 2s_*$, then for all $s_0 \le s \le \frac{s_0}{2}$, we have

$$\begin{aligned} N_r^2(Z(.,s)) &\le N_r^2(S_\sigma(s-s_0)Z(.,s_0)) + C\textstyle\int_{s_0}^s N_r^2(S_\sigma(s-\tau)(Z(.,\tau)^2))d\tau \\ &\quad + C\int_{s_0}^s N_r^2(S(s-\tau)(e^\tau))d\tau \\ &\quad + C\int_{s_0}^s N_r^2(S_\sigma(s-\tau)((1+e^{-\tau/2})\chi_{\varepsilon_0}(.,\tau)))d\tau \\ &\quad - 2\int_{s_0}^s N_r^2(S_\sigma(s-\tau)(\mathrm{div}(\nu|\partial_y\gamma)))d\tau \\ &= J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Arguing as in Part 4 and using (3.83), we prove :

Claim 3.2.23.

$$\begin{aligned} |J_1| &\le Ce^{(s-s_0)(1+\sigma)} e^{s_0(1-\varepsilon)}, \\ |J_2| &\le C\int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}} \left(N_r^2(Z(.,s)^2)\right)d\tau \\ &\quad + Ce^{(s-s_0)(1+\sigma)} e^s \text{ with } R_0 = 4\varepsilon_0, \\ |J_3| &\le Ce^{(s-s_0)(1+\sigma)} e^s(1+e^{-s/2}), \\ |J_4| &\le Ce^{(s-s_0)(1+\sigma)} e^{-\alpha e^{-s}} \text{ where } \alpha > 0, \\ |J_5| &\le Ce^{(s-s_0)(1+\sigma)} e^{-\beta e^{-s}} \text{ where } \beta > 0. \end{aligned}$$

Proof : To estimate $J_{i=1,3,4}$, see page 1584 in [Vel92]. To treat J_2 and J_5, we proceed as in the proof of Lemma 3.2.14 of the previous part.■
Summing up $J_{i=1..5}$, we obtain :

$$N_r^2(Z(.,s)) \leq \\ Ce^{(s-s_0)(1+\sigma)}e^{(1-\varepsilon)s} + C\int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}}\big(N_r^2(Z(.,s)^2)\big)d\tau,$$

then using Proposition 3.2.17, we get $N^2_{r(s,s_0)}(Z(.,s)) \leq Ce^{(s-s_0)(1+\sigma)}e^{(1-\varepsilon)s}$ as $s \to -\infty$ for $s_0 \leq s \leq \frac{s_0}{2}$. If we fix $s = s_0/2$, then we obtain $N^2_{r(s,s_0)}(Z(.,s)) \leq Ce^{s(2(1-\varepsilon)-(1+\sigma))} \leq Ce^{s(1-(2\varepsilon+\sigma))} \to 0$ as $s \to -\infty$, since ϵ is small enough and $\sigma = \frac{1}{100}$. This concludes the proof of Lemma 3.2.21.■
As announced earlier, we don't give the proof of Lemma 3.2.22 and refer the reader to Part 4 and Section 2 from [Vel92]. This concludes the proof of Proposition 3.2.19 and rules out case (ii) of Proposition 3.2.7.

Conclusion of Part 3 and the proof of the Liouville theorem
We conclude from Part 4 and 5 that cases (ii) and (iii) of Proposition 3.2.7 are ruled out. By Part 3, we obtain that $w \equiv \kappa$ or $w \equiv \varphi(s - s_0)$ for some real s_0, where φ is defined in (3.13), which is the desired conclusion of Theorem 4.

3.3 Appendix

3.3.1 Proof of Proposition 3.2.7

We proceed as in Appendix A [MZ98a], the most important difference, is that in our case we have not w bounded. Let us introduce some notations,

$$v_+(y,s) = v_0(s)h_0(y) + v_1(s)h_1(y), \quad v_{null}(y,s) = v_2(s)h_2(y).$$

We divide the proof in two parts : in part 1, we show that either v_{null} or v_+ is predominant in L^2_ρ as $s \to -\infty$. In part 2, we show that in the case where v_+ is predominant, then either $v_0(s)$ or $v_1(s)$ predominates the other.

Step 1 of the Proof : Competition between v_+, v_{null} and v_-
First, we recall from (3.38)

$$\forall (y,s) \in \mathbb{R}^2,\ \partial_s v = \mathcal{L}v - f(v). \tag{3.88}$$

Let us introduce some notations

$$z(s) = \|v_+(.,s)\|_{L^2_\rho} \quad , x(s) = \|v_{null}(.,s)\|_{L^2_\rho} \text{ and } y(s) = \|v_-(.,s)\|_{L^2_\rho}.$$

Projection (3.88) onto the unstable subspace of $\mathcal{L}$ forming the L^2_ρ-inner product with v_+, and using standard inequalities, Lemma 3.2.6 and (3.40). We get

$$\dot{z} \geq \frac{1}{2}z - N, \text{ where } N = \|v^2\|_{L^2_\rho}.$$

Working similarly with v_{null} and v_- we arrive at the system

$$\begin{array}{lcl} \dot{z} & \geq & \frac{1}{2}z - N \\ |\dot{x}| & \leq & N, \\ \dot{y} & \leq & -\frac{1}{2}y + N. \end{array} \tag{3.89}$$

If we knew for $|s|$ large enough

$$N \leq \varepsilon(x + y + z), \tag{3.90}$$

we could use ODE techniques to conclude the step. However, we do not have this information at this stage. We thus estimate N proceeding as in Section 4 in [FK92b]. We note that in [FK92b] (3.90) was proved under the additional assumption that $v(y, s)$ is uniformly bounded. One can check in [FK92b] that this assumption is only used for the derivation of the estimate $|f(v)| \leq c|v|$, which in our case is true by (3.40). Then, we get by (3.90) and (3.89),

$$\begin{array}{lcl} \dot{z} & \geq & \left(\frac{1}{2} - \varepsilon\right) z - \varepsilon(x + y) \\ |\dot{x}| & \leq & \varepsilon(x + y + z), \\ \dot{y} & \leq & -\left(\frac{1}{2} + \varepsilon\right) y + \varepsilon(x + z). \end{array} \tag{3.91}$$

Now, let us recall Lemma A.1 from [NZ08].

Lemma 3.3.1. *(Merle-Zaag) Let $x(s)$, $y(s)$, and $z(s)$ be absolutely continuous, real-valued functions that are nonnegative and satisfy*
(i)$(x, y, z)(s) \to 0$ as $s \to \infty$, and $\forall s \leq s^\star$, $x(s) + y(s) + z(s) \neq 0$, and
(ii)$\forall \varepsilon > 0$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$

$$\left\{ \begin{array}{lcl} \dot{z} & \geq & c_0 z - \varepsilon(x + y) \\ |\dot{x}| & \leq & \varepsilon(x + y + z) \\ \dot{y} & \leq & -c_0 y + \varepsilon(x + y) \end{array} \right. \tag{3.92}$$

Then, either $x + y = o(z)$ or $y + z = o(x)$ as $s \to -\infty$.

Applying Lemma above to (3.91), we get either

$$\|v_-(., s)\|_{L^2_\rho} + \|v_+(., s)\|_{L^2_\rho} = o(\|v_{null}(., s)\|_{L^2_\rho}),$$

or

$$\|v_-(s)\|_{L^2_\rho} + \|v_{null}(., s)\|_{L^2_\rho} = o(\|v_+(., s)\|_{L^2_\rho}).$$

Step 2 of the Proof : Competition between v_0 and v_1

In this step we focus on the case where $\|v_-(., s)\|_{L^2_\rho} + \|v_{null}(., s)\|_{L^2_\rho} = o(\|v_+(., s)\|_{L^2_\rho})$. We will show that it leads to either case (ii) or (iii) of Proposition 3.2.7. We want to derive from (3.88) the equations satisfied by v_0 and v_1. For this we will estimate in the following $\int f(v) k_m(y) \rho(y) dy$, for $m = 0, 1$ where

$$k_m(y) = h_m(y)/\|h_m(y)\|^2_{L^2_\rho}.$$

Lemma 3.3.2. *There is $\delta_0 > 0$ and an integer $k' > 4$ such that for all $\delta \in (0, \delta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int v^2 |y|^{k'} \rho dy \leq c_0(k') \delta^{4-k'} z(s)^2$*

Poof : This lemma is analogous to Lemma A.3 p 175 from [MZ98a], it have been proved under the additional assumption that $v(y,s)$ is uniformly bounded. One can check that this assumption is only used for the derivation of the estimate $f(v) < Cv$, which is in our case true.■

Proceeding as in Appendix A from [MZ98a] and doing the projection of equation (3.88) respectively on $k_0(y)$ and $k_1(y)$, we obtain

$$v_0'(s) = v_0(s) - \frac{\beta}{2\kappa} z(s)^2 (1 + \alpha(s)). \tag{3.93}$$

Using the same type of calculations as for $\int v^3 \rho dy$ and Lemma 3.3.2, we can prove that $\int v^2 k_1(y) \rho dy = O(z(s)^2)$. Therefore, (3.88) yields the following vectorial equation :

$$v_1'(s) = \frac{1}{2} v_1(s) + \gamma(s) z(s)^2, \tag{3.94}$$

where γ is bounded.
Proceeding as in [MZ98a], we get :

$$\forall \varepsilon > 0,\ v_0(s) = O(e^{(1-\varepsilon)s}) \text{ and } v_1(s) = C_1 e^{\frac{s}{2}} + O(e^{(1-\varepsilon)s})$$

and

$$\forall \varepsilon > 0,\ v_0(s) = -\frac{\beta}{\kappa} |C_1|^2 s e^s (1 + o(1)) + C_0 e^s + O(e^{2(1-\varepsilon)}) \text{ as } s \to -\infty \tag{3.95}$$

Two cases then arise :

- If $C_1 \neq 0$, then $v_1(s) \equiv C_1 e^{s/2} \gg v_0(s))O(se^s)$, from (3.95). We note that applying Lemma 3.2.9 to $|v|$, we have for s large enough (and $s < 0$),

$$N^2 = \int |v(y,s)|^4 \rho(y) dy \leq C^* \|v(., s - s*)\|^2_{L^2_\rho}, \tag{3.96}$$

  for some positive $s*$ and C^*.
  Recalling (3.88) and using (3.96), we obtain $\dot{y} \leq -\frac{1}{2} y + c^* \|v(., s - s*)\|^2_{L^2_\rho} \leq -\frac{1}{2} y + ce^s$. Then, we obtain $x = \|v_{null}(., s)\|_{L^2_\rho} = O(e^s)$. We conclude that $\|v(., s) - C_1 e^{s/2} y\|_{L^2_\rho} = O(e^{s(1-\varepsilon)})$ as $s \to -\infty$, for some $\varepsilon > 0$. This is case (ii) of Proposition 3.2.7.
- If $C_1 = 0$, we obtain case (iii) of Proposition 3.2.7. Indeed, let us first improve the estimate of v. In fact, from (3.95) we have $v_0 = C_0 e^s + O(e^{3s/2})$ and from (3.94) $v_1 = O(e^{3s/2})$.
  Recalling (3.88) and using (3.96), we obtain $\dot{y} \leq -\frac{1}{2} y + c^* \|v(., s - s*)\|^2_{L^2_\rho} \leq -\frac{1}{2} y + ce^{2s}$. Then, we have $y = O(e^{3s/2})$. Similarly, we obtain that $x = O(e^{3s/2})$ and we conclude that

$$\|v(., s) - v_0(s)\|_{L^2_\rho} = O(e^{3s/2}).$$

  This is case (iii) of Proposition 3.2.7.

Step 3 : Case where v_{null} dominates

In the following, we prove that (iii) of Proposition 3.2.7 holds. First, we prove the following Lemma :

Lemma 3.3.3. *Assume that*

$$\|v_+(.,s)\|_{L^2_\rho} + \|v_-(.,s)\|_{L^2_\rho} = o(\|v_{null}(.,s)\|_{L^2_\rho}) \tag{3.97}$$

holds, then

$$v(y,s) = \frac{\kappa}{4\beta s}\left(y^2-2\right) + o(\frac{1}{s}),$$

in L^2_ρ as $s \to -\infty$.

Proof : Since $v_{null}(y,s) = v_2(s)h_2(y)$, we note that $v_2(s) = \int v k_2(y)\rho$. Using (3.41), we rewrite equation (3.88) as follows

$$v_s = \mathcal{L}v - \frac{\beta}{2\kappa}v^2 + g(v), \tag{3.98}$$

where $g(v) = O(v^3)$. Projecting equation (3.98) onto $h_2(y)$ we get

$$\begin{aligned}
\partial_s v_2 &= -\frac{\beta}{2\kappa}\int v^2 k_2(y)\rho(y) + \int g(v)k_2(y)\rho(y) \\
&= -\frac{\beta}{2\kappa}\int v_{null}^2 k_2(y)\rho(y) + \frac{\beta}{2\kappa}\int \left(v_{null}^2 - v^2\right)k_2(y)\rho(y) + \int g(v)k_2(y)\rho(y) \\
&= -\frac{\beta}{2\kappa}8v_2^2 + \frac{\beta}{2\kappa}\mathcal{E}_1 + \mathcal{E}_2,
\end{aligned} \tag{3.99}$$

where we use the fact that $\int v_{null}k_2\rho = v_2^2\int h_2^2 k_2\rho = 8v_2^2$. we next estimate $\mathcal{E}_1$ and $\mathcal{E}_2$. For this we need the following lemma

Lemma 3.3.4. *There exists $\alpha_0 > 0$ and an integer $k' > 4$ such that for all $\alpha \in (0,\alpha_0)$, there exists $s_0 \in \mathbb{R}$ such that for all $s \le s_0$,*

$$\int |v|^2|y|^{k'}\rho dy \le c_0(k')\alpha^{4-k'}\int v_{null}^2\rho dy.$$

Proof : See Lemma C.1 in [MZ98a] (page 187).■
Recalling that $v = v_+ + v_{null} + v_-$, we write on the one hand :

$$\begin{aligned}
|\mathcal{E}_1| &\le \int |v_+ + v_-| \times |v + v_{null}||k_2(y)|\rho, \\
&\le c\left(\int |v_+ + v_-|^2\rho\right)^{1/2}\left\{\left(v^2k_2^2(y)\rho\right)^{1/2} + \left(\int v_{null}^2 k_2^2\rho\right)^{1/2}\right\}.
\end{aligned}$$

We know by (3.97) $\left(\int |v_+ + v_-|^2\rho\right)^{1/2} = o(v_2)$ and

$$\left(v^2 k_2^2(y)\rho\right)^{1/2} + \left(\int v_{null}^2 k_2^2 \rho\right)^{1/2} \leq \underbrace{\left(v^2 k_2^2(y)\rho\right)^{1/2}}_{I} + c|v_2|.$$

To treat I, we have from Lemma 3.3.4

$$\int v^2 k_2^2 \rho \leq c \int |v|^2 \rho + \int |v|^2 |y|^{k'} \rho \leq c \in v^2 \rho \leq c v_2^2,$$

we conclude that $\mathcal{E}_1 = o(v_2^2)$. It remains to estimate $\mathcal{E}_1$, we consider $\alpha \in (0, \alpha_0)$ and we proceed as in Appendix C from [MZ98a], (page 189). We write for $m = 0$ or $m = 2$:

$$\begin{aligned} \int |v|^3 |y|^m \rho dy &\leq \int_{|y|\leq \alpha^{-1}} |v|^3 |y|^m \rho dy + \int_{|y|\geq \alpha^{-1}} |v|^3 |y|^m \rho, \\ &\leq \varepsilon \alpha^{-m} \int_{|y|\leq \alpha^{-1}} |v|^2 \rho dy + CM\alpha^{k'-m-1}(1+\alpha) \int_{|y|\geq \alpha^{-1}} |v|^2 |y|^{k'} \rho, \\ &\leq C\left(\varepsilon \alpha^{-m} + Mc_0(k')\alpha^{4-m}\right) \int v_{null}^2 \rho dy, \end{aligned}$$

where, we used the fact that $|v| \to 0$ as $s \to -\infty$ in $L^\infty\left(B(0, \alpha^{-1})\right)$, $|v(y,s)| \leq M(1+|y|)$, Lemma 3.3.4 and $\int |v|^2 \rho dy \leq \int v_{null}^2 \rho dy$. We can then choose ε and α such that for $s \leq s_0$, $\int |v|^3 |y|^m \rho \leq \varepsilon \int v_{null}^2$ an we get $\mathcal{E}_2 = o(v_2^2)$.
So finally, we have

$$\partial_s v_2 = -\frac{\beta}{\kappa} 4 v_2^2 + o(v_2^2).$$

Solving the above, we obtain

$$v_{null} = \frac{\kappa}{4\beta s}(1 + o(1))(y^2 - 2).$$

This concludes the proof of Lemma 3.3.3.■
In order to finish the proof of (iii) of Proposition 3.2.7, we need to refine the estimates of Lemma 3.3.3 to catch the $O(\log(|s|)/s^2)$. Recalling system (3.89) and using (3.96), we obtain,

$$\dot{y} \leq -\frac{1}{2} y + c\|v(., s - s*)\|_{L^2_\rho} \leq -\frac{1}{2} y + c\frac{1}{s^2}.$$

Then, integrating $(ye^{s/2})' \leq C\frac{e^s}{s^2}$ between $-\infty$ and s, we get $y \leq \frac{C}{s^2}$. Doing the same for $z = \|v_+(., s)\|_{L^2_\rho}$, we obtain $(ze^{-s/2})' \geq C\frac{e^s}{s^2}$, integrating between s and $s_0 \geq s$, we get $z \leq \frac{C}{s^2}$.
Proceeding as in the proof of Lemma 3.3.3, we write

$$\begin{aligned} \partial_s v_2 &= -\frac{\beta}{2\kappa} \int v_{null}^2 k_2(y)\rho(y) + \frac{\beta}{2\kappa} \int \left(v_{null}^2 - v^2\right) k_2(y)\rho(y) + \int g(v) k_2(y)\rho(y), \\ &= -\frac{\beta}{2\kappa} 8 v_2^2 + \frac{\beta}{2\kappa} \mathcal{E}_1 + \mathcal{E}_2. \end{aligned} \tag{3.100}$$

Then, we have

$$\begin{aligned}
|\mathcal{E}_1| &\leq \int |v_+ + v_-| \times |v + v_{null}||k_2(y)|\rho, \\
&\leq c\left(\int |v_+ + v_-|^2\rho\right)^{1/2} \left\{ \left(v^2 k_2^2(y)\rho\right)^{1/2} + \left(\int v_{null}^2 k_2^2 \rho\right)^{1/2} \right\}, \\
&\leq \varepsilon \left(\int v_{null}^2\rho\right)^{1/2} \left\{ c\left(\int v^4\rho\right)^{1/4} + c\left(\int v_{null}^2\rho\right)^{1/2} \right\}.
\end{aligned}$$

Using the fact that $\|v_{null}(.,s)\|_{L^2_\rho} \sim \frac{C}{s}$ and (3.96), we have

$$\int v^4\rho \leq c\left(v^2(.,s-s*)\rho\right)^2 \leq \frac{c}{(s-s*)^2} \leq \frac{c}{s^2}.$$

Thus, $\mathcal{E}_1 \leq \frac{C}{s^3}$. Similarly, we obtain $\mathcal{E}_2 \leq \frac{C}{s^3}$. Then, we have from (3.100) :

$$\partial_s v_2 = -\frac{4\beta}{\kappa}v_2^2 + O(\frac{1}{s^3}) = -\frac{4\beta}{\kappa}v_2^2\left(1 + O(\frac{1}{s})\right).$$

By integerating, we obtain the desired estimation

$$v_2(s) = \frac{\kappa}{4\beta s} + O(\log(|s|)/s^2).$$

This concludes the proof of Proposition 3.2.7.■

3.3.2 Equations of Z in Parts 4 and 5

Equation of Z in Part 4 : In this part we establish the equations (3.67) and (3.72) satisfied by Z and finishes the proof of Lemma 3.2.14 and Claim 3.2.16. If we recall from (3.66) that $\nu = w - F$, where F is defined by (3.62), then we see from (3.7) that ν satisfies the following equation for all $(y,s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0\sqrt{-s}$.

$$\partial_s \nu = \mathcal{L}\nu + l(y,s)\nu - B(\nu) + R(y,s), \tag{3.101}$$

where $\mathcal{L}$ is defined in (3.48),

$$\begin{aligned}
l(y,s) &= \left(-\tfrac{\beta}{\beta+1} + \beta F^{-(\beta+1)}\right), \\
B(\nu) &= (F+\nu)^{-\beta} - F^{-\beta} + \beta F^{-(\beta+1)}\nu, \\
R(y,s) &= -\partial_s F + \partial_y^2 F - \tfrac{1}{2}y \cdot \partial_y F + \tfrac{F}{\beta+1} - F^{-\beta}.
\end{aligned} \tag{3.102}$$

Using Taylor's formula, the fact that F is bounded for $|y| \leq 4\varepsilon_0\sqrt{-s}$ and proceeding as in the proof of Lemma 3.2.6, we readily obtain for all $s \leq s_0$ and $|y| < 4\varepsilon_0\sqrt{-s}$

$$|B(\nu)| \leq C|\nu|^2, \;\; |R(y,s)| \leq C\left(\frac{|y|^2+1}{s^2} + \chi_{\varepsilon_0}\right), \;\; |l(y,s)| \leq C\min\left[\frac{(1+|y^2|)}{|s|}, 1\right]$$

with χ_{ε_0} defined in (3.64). Therefore, we write for $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$|l(y,s)| \leq C\left\{\frac{(1+\varepsilon_0^2|s|)}{|s|} + \chi_{\varepsilon_0}\right\} \leq C\left\{2\varepsilon_0^2 + \chi_{\varepsilon_0}\right\}.$$

Using Kato's inequality, we obtain for $z = |\nu|$, $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$\partial_s z - \partial_y^2 z + \frac{1}{2} y \cdot \partial_y z - (1+\sigma) z \leq C\left(z^2 + \frac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right),$$

where we fix ε_0 small enough so that $\sigma = C\varepsilon_0^2 \leq \frac{1}{100}$ (use the fact that $\chi_{\varepsilon_0}\gamma \leq \gamma^2 + \chi_{\varepsilon_0}$), with the cut-off function γ defined by (3.65).
We define $Z = z\gamma$ and we obtain for $|s|$ large enough :

$$\begin{aligned}\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1+\sigma) Z \quad &\leq C\left(Z^2 + \tfrac{(y^2+1)}{s^2} + \chi_{\varepsilon_0}\right) \\ &+ z\left(\partial_s \gamma - \partial_y^2\gamma + \frac{y}{2}\cdot\partial_y\gamma\right) - 2\partial_y\gamma\cdot\partial_y z,\end{aligned}$$

(here, we used the fact that $\gamma z^2 = Z^2 + (\gamma - \gamma^2) z^2 \leq Z^2 + C\chi_{\varepsilon_0}$). The last terms in this equation are the cut-off terms. By (i) of Lemma 3.2.1 and (3.37), we get

$$z(y,s) \leq C(1+\sqrt{-s}) \text{ for all } |y| \leq 4\varepsilon_0\sqrt{-s},$$

then we obtain using the fact that $z\big(\partial_s\gamma - \partial_y^2\gamma + \frac{y}{2}\cdot\partial_y\gamma\big) - 2\partial_y\gamma\partial_y z \leq C(1+\sqrt{-s})\chi_{\varepsilon_0} - 2\mathrm{div}(z\partial_y\gamma)$, we obtain for $|s|$ large enough :

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1+\sigma) Z \leq C\left(Z^2 + \frac{(y^2+1)}{s^2} + (1+\sqrt{-s})\chi_{\varepsilon_0}\right) - 2\mathrm{div}\left(|\nu|\partial_y\gamma\right),$$

which is the desired equation (3.67) in Lemma 3.2.14.

In the following, we will establish inequality (3.72). Let us rewrite (3.101) as follow

$$\partial_s \nu \quad = \quad \left(\mathcal{L} - \tfrac{\beta}{\beta+1}\right)\nu - B_1(\nu) + R(y,s),$$

where

where $B_1(\nu) = (F+\nu)^{-\beta} - F^{-\beta}$ and R is defined in (3.102)

Proceeding as in the proof of Lemma 3.2.6, we get $|B(\nu)| \leq C|\nu|$. Now, we do as in the proof of (3.67), and we obtain the wanted result (3.72).

Equation of Z in Part 5 : In the following, we determine the equations satisfied by Z in Part 5. We note by $\nu = w - f$. We can see from (3.82), that ν satisfies the following equation for all $(y,s) \in \mathbb{R}\times\mathbb{R}$, such that for $|y| < 4\varepsilon_0 e^{-s/2}$

$$\partial_s \nu \quad = \quad \mathcal{L}\nu + l(\nu) - B(\nu) + R(y,s),$$

where

$$\begin{aligned} l(\nu) &= \left(-\tfrac{\beta}{\beta+1} + \beta f^{-(\beta+1)}\right)\nu = l(y,s)\nu, \\ B(\nu) &= \left((f+\nu)^{-(\beta)} - f^{-\beta} + \beta f^{-(\beta+1)}\right)\nu, \\ R(y,s) &= e^s \partial_y^2 G(y e^{s/2}). \end{aligned}$$

Using Taylor's formula, the fact that f is bounded for $|y| \leq 4\varepsilon_0 e^{s/2}$ and proceeding as in the proof of Lemma 3.2.6, we prove that for $|s|$ large and $|y| \leq 4\varepsilon_0 e^{-s/2}$

$$|B(\nu)| \leq C|\nu|^2, |R(y,s)| \leq Ce^s + \chi_{\varepsilon_0}(y,s),$$

with χ_{ε_0} is defined by (3.85). We obtain for $|y|e^{s/2} \leq 4\varepsilon_0$ and s large

$$|l(y,s)| \leq C \min\left[|y|e^{s/2}, 1\right].$$

If we consider χ_{ε_0} defined in (3.85), then, we write for $|s|$ large and $|y| \leq 4\varepsilon_0 e^{-s/2}$:

$$|l| \leq C\left\{|y|e^{s/2} + \chi_{\varepsilon_0}\right\} \leq C\left\{\varepsilon_0 + \chi_{\varepsilon_0}\right\}.$$

Using Kato's inequality, we obtain for $z = |\nu|$, $|s|$ large enough and $|y|e^s \leq 4\varepsilon_0$,

$$\partial_s z - \partial_y^2 z + \frac{1}{2} y \cdot \partial_y z - (1+\sigma) z \leq C\left(z^2 + e^s + \chi_{\varepsilon_0}\right),$$

where $\sigma = C\varepsilon_0 = \frac{1}{100}$. Now, we consider the cut-off function γ, we define $Z = z\gamma$ and we obtain for $|s|$ large :

$$\begin{aligned} &\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1+\sigma) Z \\ &\leq C\left(Z^2 + e^s + \chi_{\varepsilon_0}\right) - z\left(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma\right) + 2\partial_y \gamma \partial_y z. \end{aligned}$$

The last terms in this equation are the cut-off terms. By (i) of Lemma 3.2.1 we have $|\nu| \leq C(1 + e^{-s/2})$ .Then, using $z\big(\partial_s \gamma - \partial_y^2 \gamma + \frac{y}{2} \cdot \partial_y \gamma\big) - 2\partial_y \gamma \partial_y z \leq C\chi_{\varepsilon_0} + 2\mathrm{div}(z\partial_y \gamma)$, we obtain for $|s|$ large :

$$\partial_s Z - \partial_y^2 Z + \frac{1}{2} y \cdot \partial_y Z - (1+\sigma) Z \leq C\left(Z^2 + e^s + (1 + e^{-s/2})\chi_{\varepsilon_0}\right) - 2\mathrm{div}\left(\left(|\nu| \partial_y \gamma\right),\right.$$

which is the desired equation in (3.84).

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Deuxième partie

Équation des ondes semi-linéaire

Chapitre 4

$\mathcal{C}^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation

In *Communications in Partial Differential Equations* (to appear 2008)

$\mathcal{C}^{1,\alpha}$ regularity of the blow-up curve at non characteristic points for the one dimensional semilinear wave equation

Nejla Nouaili

We consider $u(x,t)$ a blow-up solution of $\partial_{tt}^2 u = \partial_{xx}^2 u + |u|^{p-1}u$, that blows-up in one space dimension. We consider its blow-up curve $x \to T(x)$ and $\mathcal{R}$ the set of non characteristic points. We prove that $T(x)$ is of class C^{1,μ_0} in $\mathcal{R}$.

Mathematical Subject classification : 35L05, 35L67, 35A20.
Keywords : wave equation, regularity, blow-up set.

4.1 Introduction

We consider the one dimensional semilinear wave equation

$$\begin{cases} \partial_{tt}^2 u &= \partial_{xx}^2 u + |u|^{p-1}u, \\ u(0) &= u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{4.1}$$

where $u(t) : x \in \mathbb{R} \to u(x,t) \in \mathbb{R}$, $u_0 \in H^1_{loc,u}$ an $u_1 \in L^2_{loc,u}$ with

$$\|v\|^2_{L^2_{loc,u}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2\,dx \text{ and } \|v\|^2_{H^1_{loc,u}} = \|v\|^2_{L^2_{loc,u}} + \|\nabla v\|^2_{L^2_{loc,u}}.$$

The Cauchy problem for equation (4.1) in the space $H^1_{loc,u} \times L^2_{loc,u}$ follows from the finite speed of propagation and the wellposedness in $H^1 \times L^2$ (see Ginibre, Soffer and Velo [GSV92]). The existence of blow-up solutions follows from energy techniques (see Levine [Lev74]). More blow-up results can be found in Cafarelli and Friedman [CF85], [CF86], Alinhac [Ali95], [Ali02] and Kichenassamy and Littman [KL93b], [KL93a].

Consider u is a blow-up solution of (4.1). We define a continuous curve Γ as the graph of a function $x \to T(x)$ such that u cannot be extended beyond the set

$$D_u = \left\{(x,t)\big|t < T(x)\right\}. \tag{4.2}$$

The set D_u is called the maximal influence domain of u, according to the terminology introduced by Alinhac [Ali95]. From the finite speed of propagation, T is a $1-$Lipschitz function (See Alinhac [Ali95]). Let $\bar{T}$ be the infinimum of $T(x)$ for all $x \in \mathbb{R}$. The time $\bar{T}$ and the surface Γ are called (respectively) the blow-up time and the blow-up surface of u. A point $x_0 \in \mathbb{R}$ is called a non characteristic point if

$$\exists \delta_0 = \delta_0(x_0) \in (0,1) \text{ and } t_0(x_0) < T(x_0) \text{ such that } u \text{ is defined on } \mathcal{C}_{x_0,T(x_0),\delta_0} \cap \{t \geq t_0\} \tag{4.3}$$

where

$$\mathcal{C}_{\bar{x},\bar{t},\bar{\delta}} = \left\{(x,t) \middle| t < \bar{t} - \bar{\delta}|x - \bar{x}|\right\}. \tag{4.4}$$

We denote by $\mathcal{R}$, the set of non characteristic point.

Given $x_0 \in \mathbb{R}$, we introduce the following self-similar change of variables :

$$w_{x_0}(y,s) = (T(x_0) - t)^{\frac{2}{p-1}} u(x,t),\ y = \frac{x - x_0}{T(x_0) - t},\ s = -\log(T(x_0) - t). \tag{4.5}$$

This change of variables transforms the backward light cone with vertex $(x_0, T(x_0))$ into the infinite cylinder $(y,s) \in B \times [-\log(T(x_0)), +\infty)$, where $B = B(0,1)$. The function w_{x_0} (we write w for simplicity) satisfies the following equation for all $y \in B$ and $s \geq -\log(T(x_0))$:

$$\partial^2_{ss} w = \mathcal{L}w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w \tag{4.6}$$

$$\text{where } \mathcal{L}w = \frac{1}{\rho} \partial_y \left(\rho(1-y^2)\partial_y w\right) \text{ and } \rho(y) = (1-y^2)^{\frac{2}{p-1}}. \tag{4.7}$$

The Lyapunov functional for equation (4.6) :

$$E(w(s)) = \int_{-1}^{1} \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}(\partial_y w)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1}|w|^{p+1}\right) \rho dy \tag{4.8}$$

is defined in

$$\mathcal{H} = \left\{ q \in H^1_{loc} \times L^2_{loc}(-1,1) \middle| \|q\|^2_{\mathcal{H}} \equiv \int_{-1}^{1} \left(q_1^2 + (q_1^{'})^2(1-y^2) + q_2^2\right) \rho dy < +\infty \right\}. \tag{4.9}$$

In [MZ07], Merle and Zaag find the behavior of $w_{x_0}(y,s)$ defined in (4.5) as $s \to \infty$ where x_0 is a non characteristic point. In [MZ08], they prove the $\mathcal{C}^1$ regularity of the blow-up set and the continuity of blow-up profile on $\mathcal{R}$. More precisely they prove this results (for i) see Theorem 1 from [MZ08], for ii) see Corollary 4 and Theorem 2 in [MZ07]) :

Blow-up profile near a non characteristic point.
(i) $\mathcal{R}$ is open and T is of class $\mathcal{C}^1$ on $\mathcal{R}$.
(ii) There exist positive $\mu_0 = \mu_0(p)$ and $C_0 = C_0(p)$ such that if $x_0 \in \mathcal{R}$, then there exists $d(x_0) \in (-1,1)$, $|\theta(x_0)| = 1$, $s_0(x_0) \geq -\log(T(x_0))$ such that for all $s \geq s_0(x_0)$,

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d(x_0), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s - s_0(x_0))}, \tag{4.10}$$

with $d(x_0) = T^{'}(x_0)$ and for all $|d| < 1$ and $|y| \leq 1$

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \text{ with } \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}. \tag{4.11}$$

Remark : The technique of [MZ07] and [MZ08] directly yield the fact that the convergence in (4.10) is locally uniform with respect to x_0. This fact is crucial in our argument. We

clearly state it in Lemma 4.2.2 below.
Remark : Caffarelli and Friedman proved that $T(x)$ it is a $\mathcal{C}^1$ function for $N \leq 3$ under restrictive conditions on initial data that ensure that for all $x \in \mathbb{R}^N$ and $t \geq 0$, $u \geq 0$ and $\partial_t u \geq (1+\delta_0)|\nabla u|$ for some $\delta_0 > 0$. In [CF85], they derived the same result in one dimension for $p \geq 3$ and initial data in $\mathcal{C}^4 \times \mathcal{C}^3(\mathbb{R})$.

From [MZ08], we felt that the particular value of μ_0 was not used in the proof, and that only the $\mathcal{C}^1$ regularity of $T(x)$ on $\mathcal{R}$ was derived there.
In this paper, we aim at improving the regularity of $T(x)$ on $\mathcal{R}$. Our idea is related to the work of Zaag [Zaa02a], [Zaa02b] and [Zaa06] on the regularity of the blow-up set for the following semilinear heat equation :

$$\partial_t u = \Delta u + |u|^{p-1}u, \tag{4.12}$$

where $u(x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $p > 1$ and $(N-2)p < N+2$.
Unlike the wave equation (4.1), the blow-up time for (4.12) is unique and the set of blow-up points S_u is a subset of $\mathbb{R}^N$. In [Zaa02a], [Zaa02b] and [Zaa06], the author uses the idea that a better asymptotic description of the solution near blow-up points, yields geometric constraints on S_u resulting in more regularity for S_u.
In this paper, we adapt this idea to the context of the wave equation to improve the regularity of $T(x)$. More precisely, we claim the following :

Theorem 6. *Consider u a solution of (4.1) and $x \to T(x)$ its blow-up curve of class $\mathcal{C}^1$. Then $T(x)$ is of class $\mathcal{C}^{1,\mu_0}$ in $\mathcal{R}$, the set of non characteristic points.*

Remark : We reasonably think that $\mu_0 \leq 1$. This comes from two different facts :
– the largest nonpositive eigenvalue of the linearized operator of the first order version of (4.6) around $(\kappa(d,0),0)$ is -1 (see page 80 in [MZ07]).
– having $\mu_0 > 1$ would imply that the blow-up set is always a flat line on the open set $\mathcal{R}$. This cannot occur because we know from Kichenassamy and Littman, [KL93b] and [KL93a] that any analytic, space like hypersurface can be a blow-up set of the equation

$$\partial_{tt}^2 u = \Delta u + e^u,$$

and we think that the same result should be true for equation (4.1), at least for some exponents.

The paper has only one section devoted to the proof of Theorem 6.

4.2 Refined regularity derived from asymptotic blow-up behavior

We divide this section in two parts. In Part 1, we write a crucial argument of [MZ08] which shows that the convergence in (4.10) is uniform with respect to x_0 in a neighborhood of a given non characteristic point. In Part 2 we give the proof of Theorem 6.

Part 1 : Uniform character of the convergence in (4.10).
Given some $X \in \mathcal{R}$, we show here that for some $\delta > 0$, $\sup_{|x_0 - X| < \delta} s_0(x_0)$ is bounded, which gives the local uniform convergence in (4.10). This uniform character comes from two fundamental facts proved in [MZ07] and [MZ08] for equation (4.6) :

- The following Liouville theorem for equation (4.6) proved in [MZ08] :
  Proposition 4.2.1. ***(Theorem 2' in [MZ08], A Liouville theorem for equation (4.6))*** *Consider $w(y,s)$ a solution to equation (4.6) defined for all $(y,s) \in (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$ such that for all $s \in \mathbb{R}$,*

$$\|w(s)\|_{H^1(-\frac{1}{\delta^*},\frac{1}{\delta^*})} + \|\partial_s w(s)\|_{L^2(-\frac{1}{\delta^*},\frac{1}{\delta^*})} \leq C^*,$$

  for some $\delta^* \in (0,1)$ and $C^* > 0$. Then, either $w \equiv 0$ or there exists $T_0 \geq 0$, $d_0 \in [-\delta^*, \delta^*]$ and $\theta_0 = \pm 1$ such that w can be extended to a function (still denoted by w) defined for all $(y,s) \in \{(y,s) | -1 - T_0 e^s < d_0 y\} \supset (-\frac{1}{\delta^*}, \frac{1}{\delta^*}) \times \mathbb{R}$ by

$$w(y,s) = \theta_0 \kappa_0 \frac{(1-d_0^2)^{\frac{1}{p-1}}}{(1 + T_0 e^s + d_0 y)^{\frac{2}{p-1}}},$$

  where κ_0 defined in (4.11).
  The Liouville theorem allowed the authors in [MZ08] to show that the convergence of $(w_X, \partial_s w_X)$ to the profile (described by (4.10)) holds in a larger set, namely in $H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})$ for some $\delta_0 > 0$. More precisely, we recall Lemma 2.2 from [MZ08].
  Lemma 4.2.2. ***(Convergence extension to a larger set)*** *For all $X \in \mathcal{R}$, there exists $\delta_0 > 0$ such that*

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(X) \begin{pmatrix} \kappa(d(X), .) \\ 0 \end{pmatrix} \right\|_{H^1 \times L^2(-\frac{1}{\delta_0}, \frac{1}{\delta_0})} \to 0 \text{ as } s \to \infty.$$

- The following trapping result for solutions of equation (4.6) :
  Proposition 4.2.3. ***(Theorem 3 in [MZ07], trapping near the set of non-zero stationary solutions of (4.6))*** *There exists positive ε_0, μ_0 and C_0 such that if $w \in C([s^*, \infty), \mathcal{H})$ for some $s^* \in \mathbb{R}$ is a solution of equation (4.6) such that*

$$\forall s \geq s^*,\ E(w(s)) \geq E(\kappa_0),$$

  and

$$\left\| \begin{pmatrix} w(s^*) \\ \partial_s w(s^*) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d^*, .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon^*,$$

  for some $d^* \in (-1,1)$, $\theta = \pm 1$ and $\varepsilon^* \in (0, \varepsilon_0]$, where $\mathcal{H}$ and its norm are defined in (4.9) and $\kappa(d,y)$ in (4.11), then there exists $d_\infty \in (-1,1)$ such that

$$|d_\infty - d^*| \leq C_0 \varepsilon^* (1 - d^{*2})$$

  and for all $s \geq s^*$:

$$\left\| \begin{pmatrix} w(s) \\ \partial_s w(s) \end{pmatrix} - \theta \begin{pmatrix} \kappa(d_\infty, .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 \varepsilon^* e^{-\mu_0 (s - s^*)}.$$

In the following, we give a new version of Lemma 2.6 from [MZ08] which allows that the convergence in (4.10) is locally uniform with respect to x_0.

Lemma 4.2.4. ***(Locally uniform convergence to the blow-up profile)*** *There exist positive $\mu_0 = \mu_0(p)$ and $C_0 = C_0(p)$ such that for all $x_0 \in \mathcal{R}$, there exist $\delta > 0$, $s^* \in \mathbb{R}$, such that for all $X \in (x_0 - \delta, x_0 + \delta)$ and $s \geq s^*$,*

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s^*)}, \tag{4.13}$$

Proof : Since x_0 is non characteristic, we have from Lemma 4.2.2,

$$\forall s \geq s_1,\ \|(w_{x_0}(s), \partial_s w_{x_0}(s))\|_{H^1 \times L^2(-\frac{1}{\delta_0'}, \frac{1}{\delta_0'})} \leq K \tag{4.14}$$

for some constant K, $s_1 \in \mathbb{R}$ and $\delta_0' \in (\delta_0, 1)$ is fixed. Again from the fact that x_0 is a non characteristic point, we note that (4.10) holds, hence $(w_{x_0}(s), \partial_s w_{x_0}(s))$ converges to $\theta(x_0)(\kappa(d(x_0), .), 0)$ as $s \to \infty$ in the norm of $\mathcal{H}$.
Since for a fixed $s \geq s_1$, we have $(w_X(y,s), \partial_s w_X(y,s)) \to (w_{x_0}(y,s), \partial_s w_{x_0}(y,s))$ in $\mathcal{H}$ as $X \to x_0$, from (4.14) and the continuity of solutions to equation (4.6) with respect to initial data, we know that for all $\varepsilon > 0$, there exists $s_2(\varepsilon) \geq s_1$ and $\delta(\varepsilon)$ such that for all $X \in (x_0 - \delta, x_0 + \delta)$,

$$\left\| \begin{pmatrix} w_X(s_2) \\ \partial_s w_X(s_2) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq \varepsilon.$$

Using the convergence (4.10) at the point X, the continuity and the monotonicity of the Lyapunov function (4.8), we see that

$$\forall s \geq s_2(\varepsilon),\ E(w_X(s)) \geq E(\kappa(T'(X), .)).$$

Since for all $d \in (-1, 1)$, $E(\kappa(d, .)) = E(\kappa_0)$ (see Proposition 1 from [MZ07]), we apply Proposition 4.2.3 to get for all $X \in (x_0 - \delta, x_0 + \delta)$, for all $s \geq s_2$,

$$\left\| \begin{pmatrix} w_X(s) \\ \partial_s w_X(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(T'(X), .) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \leq C_0 e^{-\mu_0(s-s_2)}.$$

This concludes the proof of Lemma 4.2.4.■

Part 2 : Proof of Theorem 6

This Part is devoted to the proof of Theorem 6. The starting point is Lemma 4.2.4 proved in [MZ07], which we translate back in the variables $u(x,t)$ in the following :

Corollary 4.2.5. ***(Corollary of Lemma 4.2.4)*** *There exists μ_0 and C_0 such that for all $x_0 \in \mathcal{R}$, there exist $\delta > 0$, $0 < t^*(x_0) < \inf_{|X-x_0|\leq\delta} T(X)$, such that for all $X \in (x_0 - \delta, x_0 + \delta)$ and $t \in [t^*(x_0), T(X))$,*

$$\sup_{|\xi - X| \leq \frac{3}{4}(T(X)-t)} \left| u(\xi,t) - \theta(x_0)\kappa_0 \frac{(1 - T'(X)^2)^{\frac{1}{p-1}}}{(T(X) - t + T'(X)(\xi - X))^{\frac{2}{p-1}}} \right| \leq C(T(X) - t)^{\mu_0 - \frac{2}{p-1}}. \tag{4.15}$$

Proof of Theorem 6 : We fix $x_0 \in \mathcal{R}$ and consider an arbitrary $\sigma \geq \frac{4}{3}$. For any $x \in (x_0 - \delta, x_0 + \delta)$, where $\delta > 0$, we define $t = t(x, \sigma)$ by :

$$\frac{|x_0 - x|}{T(x_0) - t} = \frac{1}{\sigma}. \tag{4.16}$$

Note that (x, t) is on the edge of the backward cone with vertex $(x_0, T(x_0))$ and slope σ. We also note that

$$t \to T(x_0) \text{ as } x \to x_0. \tag{4.17}$$

This is our idea : using (4.15), we get 2 different main terms for $u(x,t)$, depending on the choice of X and ξ. These two terms have to agree up to error terms, which yields constraints on $T(x)$ and $T'(x)$ leading to more regularity.
Since from (4.17), we have $t = t(x, \delta) \geq t^*(x_0)$ defined in Corollary 4.2.5 for x close enough to x_0, we are able to apply this corollary, first with $X = \xi = x$, and then with $X = x_0$ and $\xi = x$.
Using (4.15) with $X = \xi = x$, we get on the one hand :

$$\left| u(x,t) - \theta\kappa_0 \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{(T(x) - t)^{\frac{2}{p-1}}} \right| \leq C\,(T(x) - t)^{\mu_0 - \frac{2}{p-1}}. \tag{4.18}$$

On the other hand, using (4.15) with $X = x_0$ and $\xi = x$, we get

$$\left| u(x,t) - \theta\kappa_0 \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T(x_0) - t + T'(x_0)(x - x_0))^{\frac{2}{p-1}}} \right| \leq C\,(T(x_0) - t)^{\mu_0 - \frac{2}{p-1}}. \tag{4.19}$$

Since we have from (4.16) $t = T(x_0) - \sigma|x_0 - x|$, hence

$$T(x_0) - t = \sigma|x - x_0| \text{ and } (\sigma - 1)|x - x_0| \leq T(x) - t \leq (\sigma + 1)|x - x_0|,$$

we derive from (4.18) and (4.19),

$$\left| \frac{(1 - T'(x_0)^2)^{\frac{1}{p-1}}}{(T'(x_0)sign(x - x_0) + \sigma)^{\frac{2}{p-1}}} - \frac{(1 - T'(x)^2)^{\frac{1}{p-1}}}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^{\frac{2}{p-1}}} \right| \leq C|x_0 - x|^{\mu_0},$$

hence

$$\left| \frac{1 - T'(x_0)^2}{(T'(x_0)sign(x - x_0) + \sigma)^2} - \frac{1 - T'(x)^2}{\left(\frac{T(x) - T(x_0)}{|x - x_0|} + \sigma\right)^2} \right| \leq C|x_0 - x|^{\mu_0}, \tag{4.20}$$

where $sign(X) = \frac{X}{|X|}$ for $X \neq 0$.
Now, if we introduce :

$$f(\xi) = T(\xi + x_0) - T(x_0) - \xi T'(x_0) \text{ and } \lambda = T'(x_0) \tag{4.21}$$

then we have $f(0) = f'(0) = 0$ and (4.20) becomes :

$$\left| \frac{1-\lambda^2}{(\lambda sign(\xi)+\sigma)^2} - \frac{1-(f'(\xi)^+\lambda)^2}{\left(\frac{f(\xi)}{|\xi|}+\lambda sign(\xi)+\sigma\right)^2} \right| \leq C|\xi|^{\mu_0}. \tag{4.22}$$

Therefore,

$$\left| (1-\lambda^2)\left(\frac{f(\xi)}{|\xi|}+\lambda sign(\xi)+\sigma\right)^2 - (1-(f'(\xi)+\lambda)^2)(\lambda sign(\xi)+\sigma)^2 \right| \leq C|\xi|^{\mu_0},$$

and

$$\left| \left(\frac{f(\xi)}{\xi}\right)^2 \frac{1-\lambda^2}{2(\lambda sign(\xi)+\sigma)^2} + \frac{f(\xi)}{\xi}\frac{1-\lambda^2}{\lambda sign(\xi)+\sigma} + f'(\xi)^2/2 + f'(\xi)\lambda \right| \leq C|\xi|^{\mu_0}. \tag{E_σ}$$

Now, we consider two different values of σ, σ_1 and σ_2 and we take the difference between (E_σ, $\sigma=\sigma_1$) and (E_σ, $\sigma=\sigma_2$) to get :

$$\begin{aligned} &\Bigg| \left(\frac{f(\xi)}{\xi}\right)^2 \left[\frac{1-\lambda^2}{2(\lambda sign(\xi)+\sigma_1)^2} - \frac{1-\lambda^2}{2(\lambda sign(\xi)+\sigma_2)^2}\right] \\ &+ \frac{f(\xi)}{\xi}\left[\frac{1-\lambda^2}{\lambda sign(\xi)+\sigma_1} - \frac{1-\lambda^2}{\lambda sign(\xi)+\sigma_2}\right] \Bigg| \leq C|\xi|^{\mu_0}. \end{aligned}$$

Since $\frac{f(\xi)}{\xi} \to f'(0) = 0$ as $\xi \to 0$, this yields

$$\left|\frac{f(\xi)}{\xi}\right| \leq C|\xi|^{\mu_0}. \tag{4.23}$$

Using this last estimate in equation (E_σ), this gives the following :

$$\left|f'(\xi)^2 + f'(\xi)2\lambda\right| \leq C|\xi|^{\mu_0}. \tag{4.24}$$

At this level of the proof we distinguish two cases :

Case 1 : $\lambda \neq 0$**.** Since $f'(\xi) \to f'(0) = 0$ as $\xi \to 0$, we get immediately from (4.24) $|f'(\xi)| \leq C|\xi|^{\mu_0}$. Using back the change of variables (4.21), we see that T is $\mathcal{C}^{1,\mu_0}$ near x_0.

Case 2 : $\lambda = 0$**.** We note first that by (4.24), we have :

$$|f'(\xi)| \leq C|\xi|^{\mu_0/2}. \tag{4.25}$$

Switching x and x_0 in (4.20), we get :

$$\left| \frac{1-T'(x)^2}{(T'(x)sign(x_0-x)+\sigma)^2} - \frac{1-T'(x_0)^2}{\left(\frac{T(x_0)-T(x)}{|x-x_0|}+\sigma\right)^2} \right| \leq C|x_0-x|^{\mu_0}.$$

Using the change of variable (4.21) and the same arguments used to obtain (E_σ), we have after simplifications :

$$\left|(1 - f'(\xi)^2)\left(\sigma - \frac{f(\xi)}{|\xi|}\right)^2 - (\sigma - f'(\xi)sign(\xi))^2\right| \leq C|\xi|^{\mu_0}.$$

Using (4.23) and (4.25) in the last inequality, we obtain $|f'(\xi)| \leq C|\xi|^{\mu_0}$. Using again (4.21), we see that $T(x)$ is $\mathcal{C}^{1,\mu_0}$ near x_0. This concludes the proof of Theorem 6.■

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Résumé

Théorèmes de Liouville et singularités dans les équations aux dérivées partielles.

Cette thèse est consacrée à l'étude de la formation de singularités en temps fini dans les équations semi-linéaires de la chaleur et des ondes par l'approche des Théorèmes de Liouville.

La première partie est consacrée aux équations semi-linéaires de type chaleur. Le chapitre 1 est consacré à une redémonstration simple dans le cas positif du théorème de Liouville que Merle et Zaag ont démontré pour la non-linéarité en puissance souscritique.

Nous montrons ensuite dans le deuxième et le troisième chapitres deux Théorèmes de Liouville pour une équation de la chaleur complexe sans structure de gradient pour le chapitre 2, et pour une équation avec absorption dans le chapitre 3. Nous obtenons également une propriété de localisation de ces équations qui permet de les comparer de façon précise aux solutions des équations différentielles associées.

Dans une deuxième partie, nous étudions l'équation des ondes semi-linéaire. En utilisant un Théorème de Liouville, nous parvenons à améliorer la régularité de l'ensemble d'explosion au voisinage des points caractéristiques.

Abstract

Liouville Theorems and singularities in Partial Differential Equations.

The purpose of this dissertation is the study of the formation of singularities in heat and wave equations using Liouville Theorems.

The first part is dedicated to the study of semilinear heat equations. In chapter 1, we give a simple proof in the positive case of the Liouville Theorem which was proved by Merle and Zaag in the subcritical case.

In the next two chapters, we prove two Liouville Theorems for a complex heat equation with no gradient structure in chapter 2 and for a heat equation with absorption in chapter 3. These theorems give a localization property of the equations which yield a precise comparison with the solution of the associated differential equations.

In the second part, we study the semilinear wave equation. We use a Liouville Theorem in order to get a better regularity on the blow-up set near characteristic points.