Course notes : Mathematical Tools

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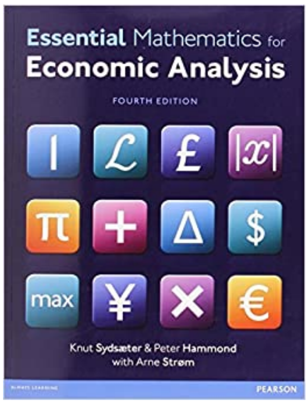
Contents

This document is taken from a mathematical upgrade and tools course. He owes a lot

- to the handout of Denis Pasquignon.
- to the following book (in English and translated into French)



There are definitely a lot of mistakes and typos left. Please do not hesitate to report them to us.

Chapter 1

Study of functions

1.1 Introduction

Functions are important in practically every area of pure and applied mathematics, including mathematics applied to economics. The language of economic analysis is full of terms like demand and supply functions, cost functions, production functions, consumption functions, etc.

1.2 Domain of Definition and Even/Odd Functions

Definition 1.2.1 *An* application f *is given by an* initial set*, a* final set*, and a* rule *which associates to all element x of the initial set a unique element of the final set, denoted $f(x)$ and called image de x through f. The rule is denoted by $x \mapsto f(x)$.*
The domain of definition of a function is the set of real numbers x for which the calculation of $f(x)$ is possible.
The graph of a function f is the set of points $(x, f(x))$ where x described the domain of definition of f.

Definition 1.2.2 *If the domain of definition D of a function f is symmetric, i.e. if we have*

$$\forall x \in I\!R, \quad x \in D \Longrightarrow -x \in D,$$

We say that f is an even function if

$$\forall x \in D, \quad f(x) = f(-x).$$

In this case, the graph of f is symmetric with respect to the y-axis.
we say that f is an odd function if

$$\forall x \in D, \quad f(x) = -f(-x).$$

In this case, the graph of f is symmetric with respect to the origin.

For example, function $f(x) = x^2$ is even for its domain $I\!R$ whereas $f(x) = x^3$ is odd for $I\!R$.

1.3 Operations on Functions

Let two functions f and g be defined on the same domain I, we can then add, subtract, multiply these two functions, therefore creating new functions $f+g$, $f-g$, fg defined on I by

$$\forall x \in I,\ \ (f+g)(x) = f(x)+g(x),\ \ (f-g)(x) = f(x)-g(x),\ \text{and}\ (fg)(x) = f(x)g(x).$$

In order to divide f by g, furthermore, it is necessary to assume that g does not become void on the domain I, if this is the case, we have

$$\forall x \in I,\ \ (\frac{f}{g})(x) = \frac{f(x)}{g(x)}.$$

Example 1.3.1 *Let*

$$\forall x \in I\!R,\ \ f(x) = x^2+1 \text{ and } g(x) = x^2-1,$$

f and g are defined on $I\!R$, we have

$$\forall x \in I\!R,\ \ (f+g)(x) = 2x^2,\ \ (f-g)(x) = 2 \text{ and } (fg)(x) = x^4-1.$$

For division, as function g becomes void on 1 and -1, we have

$$\forall x \in I\!R \setminus \{-1,1\},\ \ (f/g)(x) = \frac{x^2+1}{x^2-1}.$$

We can also have a composition of functions: this way, $f \circ g$ is the function defined on I by

$$\forall x \in I,\ \ (f \circ g)(x) = f(g(x)).$$

This means that we start by calculating $g(x)$ then we calculate the value of f for the real number $g(x)$. In order for the calculation to be possible, real number $g(x)$ needs to be part of the domain of definition of f. Let's provide two examples:

Example 1.3.2 *Let*

$$\forall x \in I\!R,\ \ f(x) = x^2+1 \text{ and } g(x) = x^2-1,$$

f and g are defined on $I\!R$, we have

$$\forall x \in I\!R,\ \ (f \circ g)(x) = f(g(x)) = (g(x))^2+1 = (x^2-1)^2+1 = x^4-2x^2+2.$$

Whereas $f \circ g$ and $g \circ f$ are two different functions:

$$\forall x \in I\!R,\ \ (g \circ f)(x) = g(f(x)) = (f(x))^2-1 = (x^2+1)^2-1 = x^4+2x^2.$$

1.4 Calculation of Derivatives

1.4.1 Definition

Definition 1.4.1 *Let a be a real number and let f be a function defined on a set D which includes a. We name rate of change of f at a, written $\theta_a(x)$, the quotient function defined by*

$$\forall x \in D \setminus \{a\},\ \ \theta_a(x) = \frac{f(x)-f(a)}{x-a}.$$

The function f is differentiable at a if the function θ_a has a limit when x approaches a. We write this limit as $f'(a)$.
we say that f is differentiable on D if f is differentiable at each point of D and we denote by f' the derivative function of f.

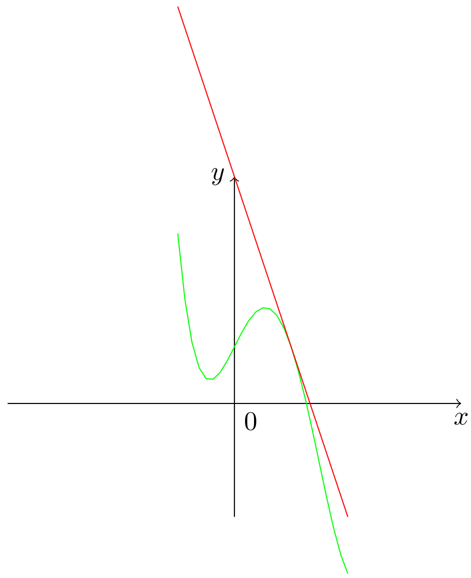
1.4.2 Equation of Tangent

Definition 1.4.2 *Let a be a real number and let f be a function defined on a set D which includes a. We assume that function f is differentiable at a, the tangent to the graph of f at point $A = (a, f(a))$ is the line passing through point A , with direction vector $(1, f'(a))$.*
An equation of tangent is

$$y = f'(a)(x - a) + f(a).$$

$f'(a)$ is the gradient of the tangent.
We say that f is differentiable on D if f is differentiable at each point of D and we denote by f' the derivative function of f.

Example 1.4.3 *In the figure below, we have traced the graph of function $f(x) = x^4 - 3x^3 + 2x + 1$ as well as the tangent at $x = 1$ with equation $y = -3x + 4$.*



1.4.3 Formulas

Function f	Derivative f'	Interval
x^n, $n \in \mathbb{N}$	nx^{n-1}	$\mathbb{R}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$	$]0, +\infty[$
$\dfrac{1}{x}$	$\dfrac{-1}{x^2}$	$\mathbb{R}^*$
$\ln x$	$\dfrac{1}{x}$	$]0, +\infty[$
e^x	e^x	$\mathbb{R}$
x^α, $\alpha \in \mathbb{R}$	$\alpha x^{\alpha-1}$	$]0, +\infty[.$

Function f	Derivative f'
$u + v$	$u' + v'$
uv	$u'v + uv'$
$\dfrac{1}{u}$	$\dfrac{-u'}{u^2}$
$\dfrac{u}{v}$	$\dfrac{u'v - uv'1}{v^2}$
$\ln(\|u\|).$	$\dfrac{u'}{u}$
e^u	$e^u \times u'$
$(u)^\alpha$	$\alpha\,(u)^{\alpha-1} \times u'$
$\sqrt{u} = (u)^{1/2}$	$\frac{u'}{2\sqrt{u}}$

1.5 Convex or Concave Functions

Definition 1.5.1 convexity
Let f be a function of class C^2 on an open interval $I =]a, b[$, that is, a function twice differentiable on I , whose second derivative is continuous on I, We say that f is convex on I if for all real numbers $x \in I$, $f''(x) \geq 0$. Likewise, we say that f is concave on I if for all real numbers $x \in I$, $f''(x) \leq 0$.

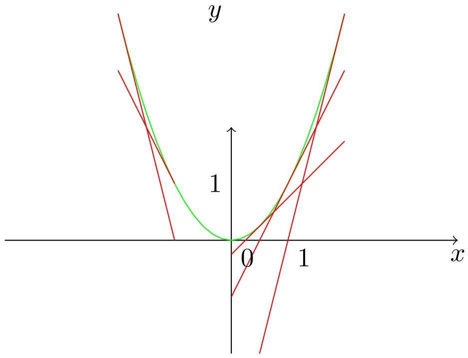
Proposition 1.5.2 *Let f be a convex function on an open interval $I =]a, b[$, then*

$$\forall c \in I,\ \forall x \in I,\quad f(x) \geq f(c) + f'(c)(x - c).$$

This inequality shows that the graph of f is above all its tangents when f is convex. likewise, let f be a concave function on an open interval $I =]a, b[$, then

$$\forall c \in I,\ \forall x \in I,\quad f(x) \leq f(c) + f'(c)(x - c).$$

Function $f(x) = x^2$ is convex on $I\!R$, we traced 4 tangents to the graph of this function, we can observe that the graph is above each tangent.

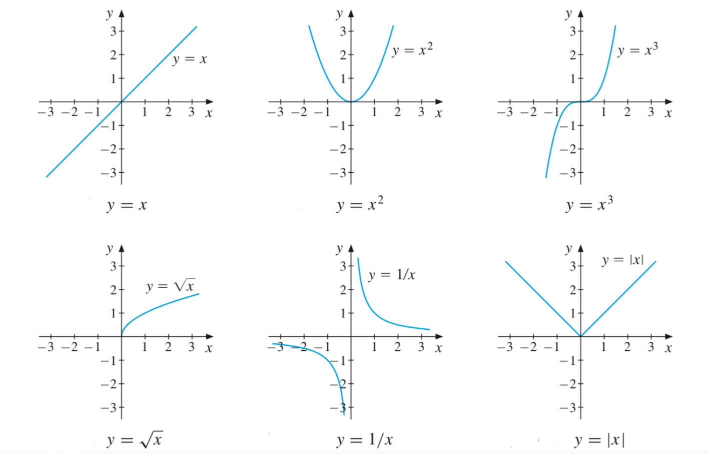


Remarque: The graphical representation of a convex function over an interval has this behavior: ∪, and that of a concave function has this one: ∩.

1.6 Methodology for the study of a function.

The study of a function needs many steps you have to know. Above we give the steps to study a function f and re-enter its graph:

- ⋆ Compute **the definition domain**.
- ⋆ Verify if the function is odd or even.
- ⋆ Study the derivability of the function f on his domain of definition.
- ⋆ If the function is differentiable,**compute its derivative function f'**.
- ⋆ Determine the sign of the derivative function to know **the variations of the function**.
- ⋆ Calculate the limits on the boundary of the definiton domain.
- ⋆ Represent **the variation table**.
- ⋆ Draw the graph of the function.



Représentation graphique des quelques fonctions usuelles.

Remark: the absolute value function $y = |x|$ is given by the definition 2.1.8, page 12.

1.7 Bijection

Definition 1.7.1 bijection
Let I be an interval of $\mathbb{R}$ and f a function defined on I. We say that f is a bijection from I to $f(I)$ if for all $y \in f(I)$, the equation $y = f(x)$ with unknown x admits a unique solution $x \in I$. If f is a bijection of I on $f(I)$, we can define the inverse function of f denoted f^{-1} defined on $f(I)$ by: for all $y \in f(I)$, $f^{-1}(y)$ is the unique solution on I of the equation $y = f(x)$.

Proposition 1.7.2 $\ln$ **is a bijection from** $\mathbb{R}^{+*}$ **to** $\mathbb{R}$, *the exponential function is the inverse application of* $\ln$

$$\forall x > 0, \ \ \forall y \in \mathbb{R}, \ \ \ln(x) = y \Leftrightarrow x = e^y.$$

Proposition 1.7.3 *Let f be a continuous function and strictly monotonic on an interval I, then f realises a bijection from I to $f(I)$.*

Graphics of f and its inverse f^{-1} are symmetric with respect to the line x=y.

1.8 Exercises

Exercise 1.8.1 1. We consider the functions f et g defined by : $f(x) = \dfrac{4x}{x+1}$ et $g(x) = \dfrac{2x-1}{x}$.

(a) Determine the definition domain of f et de g.

(b) Determine the function : $\dfrac{f}{g}$, and the definition domain.

2. We consider the functions h_1 et h_2 defined by : $h_1(x) = \sqrt{x^2-1}$ and $h_2(x) = x\ \ln(x)$. Determine the definition domain of h_1 and h_2.

Exercise 1.8.2 ***Computation of derivative function***

Calculate the derivative function of the following function

$$\begin{array}{ll} f_1(x) = \ln(x+1) \text{ sur }]-1,+\infty[, & f_4(x) = -3x^3 + x^2 - x + 17 \text{ sur } \mathbb{R}, \\ f_2((x) = \sqrt{x^2+1} \text{ sur } \mathbb{R}, & f_5(x) = x\sqrt{x+1} \text{ sur }]-1,+\infty[, \\ f_3((x) = \exp(x^3) \text{ sur } \mathbb{R}, & f_6(x) = \dfrac{x-1}{x^2+1} \text{ sur } \mathbb{R}. \end{array}$$

Exercise 1.8.3 Let $C(x)$ be the total cost of producing x units of a good. The quotient $F(x) = \frac{C(x)}{x}$ is the average cost of production when x units are produced. Calculate $F'(x)$.

.

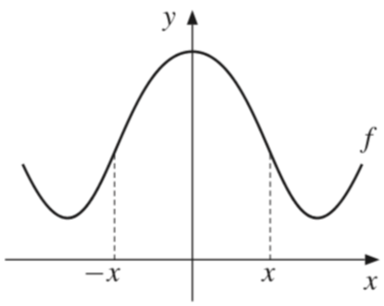
Exercise 1.8.4 ***convex or concave functions***

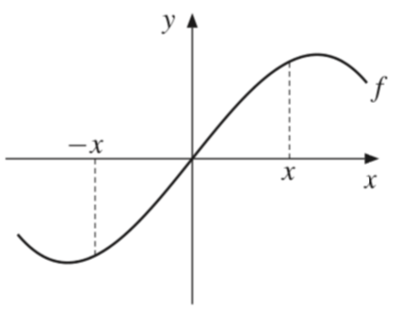
Say if the following functions are concave or convex. You have to justify.

$$\begin{array}{ll} f(x) = \sqrt{x} & g(x) = \ln(x), \\ (x) = e^{x-1} - x. & \end{array}$$

Exercise 1.8.5 ***odd and even functions.***

Say if the following graphs correspond to an odd or even function, and justify it.





Exercise 1.8.6 Let's consider function g defined on $I\!R$ by $g(x) = \frac{e^x}{e^x+1}$.

1. Demonstrate, justifying the inequalities, that

$$\forall x \in \mathbb{R}, \ \ 0 < g(x) < 1.$$

2. Demonstrate that g is a bijection from $\mathbb{R}$ to $]0,1[$.

3. Trace the graph of g.

4. Determine $g^{-1}(y)$ for $y \in]0,1[$.

Chapter 2

Line equation

2.1 Linear Function

2.1.1 Definition and Properties

Definition 2.1.1 Linear function *A linear function f is defined by*

$$\forall x \in \mathbb{R}, \quad f(x) = ax + b \text{ with } a \in \mathbb{R} \text{ and } b \in \mathbb{R}.$$

The curve representing f is a line of the plane and the equation of this line is

$$y = ax + b.$$

Every linear function is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \quad f'(x) = a.$$

The real number a is called the line's gradient or the slope of the line.

Remark:

$$\lim_{x \to +\infty} f(x) = \begin{cases} +\infty \text{ if } a > 0 \\ -\infty \text{ if } a < 0 \end{cases} \qquad \lim_{x \to -\infty} f(x) = \begin{cases} -\infty \text{ if } a > 0 \\ +\infty \text{ if } a < 0 \end{cases}.$$

Proposition 2.1.2 Calculation of the gradient
Let $A = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$ and $B = \begin{pmatrix} x_B \\ y_B \end{pmatrix}$ be two different points of a line $\mathcal{D}$ of $\mathbb{R}^2$. We assume that $x_A \neq x_B$, the gradient, denoted a, of the line $\mathcal{D}$ is

$$a = \frac{y_B - y_A}{x_B - x_A}.$$

Definition 2.1.3 Cartesian equation of a line
$\mathcal{D}$ is a line of $\mathbb{R}^2$ if and only if there exist three real numbers α, β and γ with $(\alpha, \beta) \neq (0,0)$ such that

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \,,\, \alpha x + \beta y + \gamma = 0\}.$$

"$\alpha x + \beta y + \gamma = 0$" is called cartesian equation of $\mathcal{D}$. The vector $(-\beta, \alpha)$ is a direction vector of $\mathcal{D}$.

Example 2.1.4 *Let $A = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ be two points, we look for the Cartesian equation of line (AB). We have to solve the system with two equations and three unknowns denoted by a, b and c :*

$$\begin{cases} 2\alpha + 7\beta + \gamma &= 0, \\ -\alpha + \beta + \gamma &= 0, \end{cases}$$

This system is equivalent to

$$\begin{cases} 2\alpha + 7\beta + \gamma &= 0, \\ 9\beta + 3\gamma &= 0, \end{cases} \iff \begin{cases} \alpha &= 2/3\gamma, \\ \beta &= -1/3\gamma, \end{cases}$$

We therefore have infinite solutions, we just need to determine one solution, for example for $\gamma = 3$, we have $\alpha = 2$ and $\beta = -1$. This way, an equation of the line (AB) is

$$2x - y + 3 = 0.$$

which we can write $y = 2x + 3$.

The line (AB) is the graph of function $f(x) = 2x + 3$. The gradient of this line is 2.

Example 2.1.5 *Let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be the point and $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be a vector, we are looking for the Cartesian equation of the line passing through A with direction vector $\vec{u}$. We have $-\beta = 2$ and $\alpha = 1$, we just need to determine γ which is a solution of*

$$1 + 2 + \gamma = 0 \text{ hence } \gamma = -3.$$

Consequently, an equation passing through A with direction vector $\vec{u}$ is

$$x - 2y = 3.$$

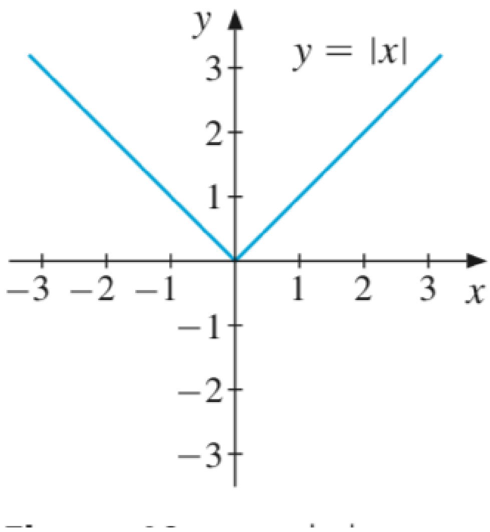
Example 2.1.6 *Let's consider the line with Cartesian equation $2x + 3y - 1 = 0$. Points $A = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ verify the equation, therefore are on the line, which is also line (AB).*

A direction vector of this line is also $\vec{u} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ but also $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$. We write $\overrightarrow{AB} = 2\vec{u}$.

Remark 2.1.7 *The cartesian equation is not unique. Equations : $x + 2y + 3 = 0$ and $4x + 8y + 12 = 0$ define the same line.*

Definition 2.1.8 Absolute Value
Let x be a real number, we note the absolute value of x $|x| = \begin{cases} x & si \quad x \geq 0, \\ -x & si \quad x < 0. \end{cases}$

We remark for any $x \in I\!R$, $|x| = |-x|$.



The graph of absolute value function.

Proposition 2.1.9 Properties of the absolute value
for any x, y real and $a > 0$,

$$\begin{cases} |x| = 0 \iff & x = 0, \text{ the only real numbre verifying } |x| = 0 \text{ is } 0. \\ |xy| & = |x||y|. \\ |x| \leq a \iff & -a \leq x \leq a. \end{cases}$$

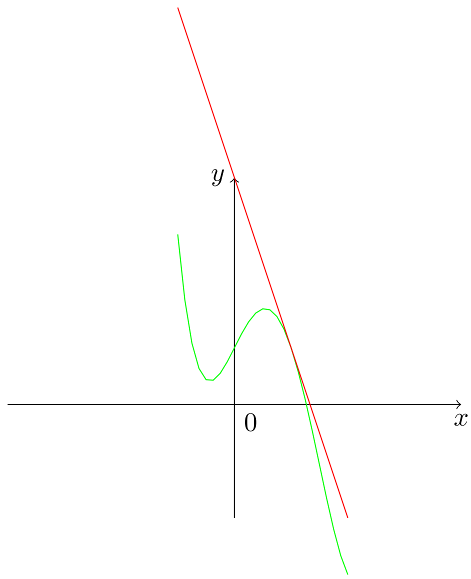
2.1.2 Equation of Tangent

Definition 2.1.10 *Let a be a real number and let f be a function defined on a set D which includes a. We assume that function f is differentiable at a, the tangent to the graph of f at point $A = (a, f(a))$ is the line passing through point A , with direction vector $(1, f'(a))$.*
An equation of tangent is

$$y = f'(a)(x - a) + f(a).$$

$f'(a)$ is the gradient of the tangent.

Example 2.1.11 *In the figure below, we have traced the graph of function $f(x) = x^4 - 3x^3 + 2x + 1$ as well as the tangent at $x = 1$ with equation $y = -3x + 4$.*

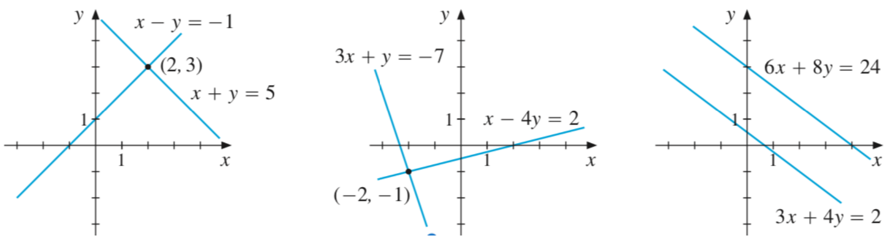


2.2 Linear system

We consider the three following linear systems:

$$(a)\left\{\begin{array}{rcl} x+y & = & 5 \\ x-y & = & -1 \end{array}\right. \quad (b)\left\{\begin{array}{rcl} 3x+y & = & -7 \\ x-4y & = & 2 \end{array}\right. \quad (c)\left\{\begin{array}{rcl} 3x+4y & = & 2 \\ 6x+8y & = & 24 \end{array}\right.$$

Below are three graphic representations corresponding to these three systems.



Graphic resolution of the systems (a), (b) et (c)

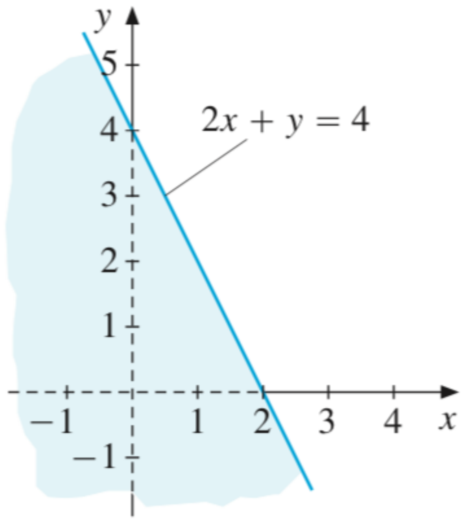
Exercise: Solve each of the systems and check your result with the graph.

2.3 Linear inequality

Represent the set of points (x, y) such that

$$2x + y \leq 4.$$

Solution : The inequality can be written $y \leq -2x + 4$.
We first represent Δ, the line with the equation $y = -2x + 4$. Then the set of points (x, y) verifying $y \leq -2x + 4$ must have values of y below the line Δ, see the figure below



Example 2.3.1 ***Example of linear functions in economy***

In economics, most linear models are approximations of more complicated models. Below is an example where the linear model (affine function) is used.

Find and interpret the slopes of the following straights lines

- *(a)* $C = 55; 73x + 182100000$*: Estimated cost function for the US Steel Corp (1917-1938). (C is the total cost in dollars per year and x is the production of steel in tons per year).*
- *(b)* $q = -0, 15p + 0, 14$*: Esimated annual demand function for rice in India for the period 1949-1964 (p is the price in Indians rupees, and q is consumption per person).*

Solution:

- *(a) The slope is* $55, 73$*, which means that if production increases by 1 ton, then the cost increases by* $55, 73$ *dollars.*
- *(b) The slope is*$-0, 15$*, which means that if the price increases by one Indian rupee, then the quantity demand decreases by 0.15 units.*

2.3.1 Exercises

Exercise 2.3.1 We are on a plane with orthonormal basis.

1. Let's consider the line D_1 passing through points $A = (2, -3)$ and $B = (4, -5)$.
   (a) Graphically represent this line.
   (b) What is the gradient of this line?
   (c) Determine a Cartesian equation of the line D_1.
   (d) Is point $O = (0, 0)$ located on the line D_1 ?
2. Let's consider the line D_2 passing through points $C = (1, 1)$ and $D = (3, 3)$.
   (a) Graphically represent this line.
   (b) What is the gradient of this line?
   (c) Determine a Cartesian equation of the line D_2.
3. Determine the intersection between D_1 and D_2.
4. Let's consider the line D_3 passing through point $E = (-1, 3)$ with direction vector $u = (-3, 2)$.
   (a) Graphically represent this line.
   (b) Determine a Cartesian equation of the line D_3.
   (c) Is point $O = (0, 0)$ located on the line D_3 ?

Exercise 2.3.2 Let's consider line

$$D = \{(x, y) \in \mathbb{R}^2,\ \ 2x + 3y = 6\}.$$

1. Determine two distinct points of D.
2. Graphically represent this line.
3. Determine a direction vector of D which will be notated as $\vec{u}$.
4. Is point $A = (1, 1)$ located on the line D ?
5. What is the cartesian equation of the line Δ passing through point A with direction vector $\vec{u}$.
6. Represent line Δ on the same graphic.

Exercise 2.3.3 Let's consider the system:

$$\begin{cases} 2x + y &= 1, \\ x - y &= 2 \end{cases}$$

1. Trace, on an orthonormal basis, equation lines $2x + y = 1$ and $x - y = 2$, and then determine on the graph the intersection point of these two lines. Deduce a solution for the system.
2. Solve the system and verify the result of the previous question.

Exercise 2.3.4 Graphically solve the following systems:

$$\begin{cases} x - y &= 5, \\ x + y &= 1. \end{cases}$$

$$\begin{cases} 3x + 4y &= 1, \\ 6x + 8y &= 6. \end{cases}$$

Exercise 2.3.5 Represent in the plane the set of all the points whose coordinates (x; y) satisfy to the three equations

$$3x + y \leq 12, \quad x - y \leq 1, \quad 3x + y \geq 3.$$

Exercise 2.3.6 ***Macroeconomic model***
At the beginning of the year, a person had a total of 10 000 € in two accounts. the interest rates were 5% et 7, 2% per year, respectively. If the person has made no transfers during the year, and has earned a total of 676 € interest. What was the initial balance in each of the two accounts ?

Chapter 3

Polynomial functions and rational fractions

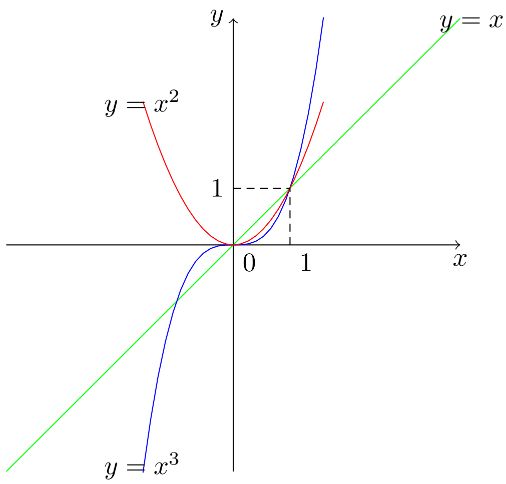
3.1 Definition and Properties

Definition 3.1.1 The integer power function *For all non-zero natural integers $n \in I\!N^*$, the power function f_n is defined by*

$$\forall x \in I\!R, \;\; f_n(x) = x^n.$$

For $n = 0$, f_0 is the constant function equal to 1.

Below the graphs of the functions $f_1(x) = x$, $f_2(x) = x^2$ et $f_3(x) = x^3$.



Proposition 3.1.2 Derivative *For all natural numbers n, the power function is differentiable on $I\!R$ and*

$$\forall x \in I\!R, \;\; \forall n \in I\!N^*, \;\; (x^n)' = nx^{n-1}$$

Example 3.1.3 $(x^2)' = 2x$ *and* $(x^3)' = 3x^2$

Definition 3.1.4 Polynomials and Rational Fractions *For all natural numbers* $n \in \mathbb{N}$*, function* P *is a polynomial if it is possible to write*

$$\forall x \in \mathbb{R}, \quad P(x) = a_0 + \cdots + a_n x^n$$

where $(a_0, \cdots, a_n) \in \mathbb{R}^{n+1}$*. If* $a_n \neq 0$*, we say that the degree of* P *is* n*.*
A rational fraction is the quotient of two polynomials.

Example 3.1.5 *Below is an example of polynomial* P

$$\forall x \in \mathbb{R}, \quad P(x) = 4 + 3x + 8x^3 \text{ and } P'(x) = 3 + 24x^2$$

and an example of rational fraction

$$\forall x \in \mathbb{R} \setminus \{1\}, \quad F(x) = \frac{2x+1}{x-1} \text{ and } F'(x) = \frac{-3}{(x-1)^2}$$

Definition 3.1.6 Factorisation of a degree 2 polynomial *Let's consider a polynomial* P *of degree 2, there exist three real numbers* $a \neq 0$ *,* b *and* c *such that*

$$\forall x \in \mathbb{R}, \quad P(x) = ax^2 + bx + c.$$

We set

$$\Delta = b^2 - 4ac.$$

- *if* $\Delta > 0$*, then* P *has two separate real roots*

  $$x_1 = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{\Delta}}{2a},$$

  and we can write P *in factored form:*

  $$\forall x \in \mathbb{R}, P(x) = a(x - x_1)(x - x_2).$$

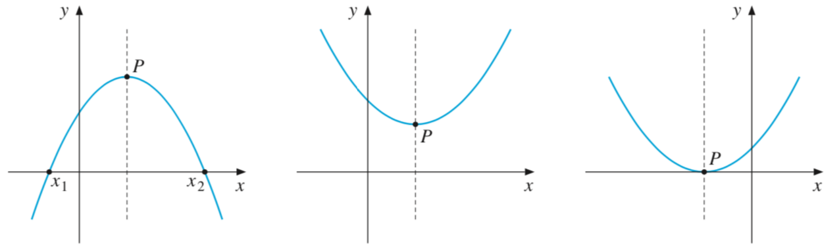
- *If* $\Delta = 0$*, then* P *has a real root*

  $$x_1 = \frac{-b}{2a},$$

  and we can write P *in factored form:*

  $$\forall x \in \mathbb{R}, P(x) = a(x - x_1)^2.$$

- *If* $\Delta < 0$*, then* P *has no real roots.*



(a) $a < 0,\ b^2 > 4ac$ (b) $a > 0,\ b^2 < 4ac$ (c) $a > 0,\ b^2 = 4ac$

Graphical representation of $P(x) = ax^2 + bx + c$ according to values a, b and c

Proposition 3.1.7 Factorisation *Let P be a polynomial. If a is a root of P, that is, $P(a) = 0$. then, there exists a polynomial Q such that*

$$P = (x - a)Q.$$

Example 3.1.8 *Let polynomial P be defined by*

$$P(x) = x^3 + 3x^2 - x - 3.$$

We write that 1 is a root of P because $P(1) = 1 + 3 - 1 - 3 = 0$, therefore there exists a polynomial Q such as

$$P = (x - 1)Q.$$

In order to determine polynomial Q, we begin by noticing that Q is of degree 2, therefore

$$Q(x) = ax^2 + bx + c,$$

then we develop

$$\begin{array}{rcl} (x-1)Q(x) & = & (x-1)(ax^2+bx+c), \\ & = & ax^3 + bx^2 + cx - ax^2 - bx - c, \\ & = & ax^3 + (b-a)x^2 + (c-b)x - c. \end{array}$$

The equality $P = (x - 1)Q$ becomes

$$x^3 + 3x^2 - x - 3 = ax^3 + (b-a)x^2 + (c-b)x - c.$$

By identification, we have the system

$$\left\{ \begin{array}{rcl} a & = & 1, \\ -a+b & = & 3, \\ -b+c & = & -1, \\ -c & = & -3. \end{array} \right.$$

From which

$$\left\{ \begin{array}{rcl} a & = & 1, \\ b & = & 4, \\ c & = & 3, \\ c & = & 3. \end{array} \right.$$

Consequently, we have

$$x^3 + 3x^2 - x - 3 = (x-1)(x^2 + 4x + 3).$$

Furthermore, the study of $x^2 + 4x + 3$ *shows that -1 and -3 are roots, therefore, we have*

$$x^3 + 3x^2 - x - 3 = (x-1)(x+1)(x+3).$$

3.2 Exercises

Exercise 3.2.1 Factor the following polynomials

$$P(x) = x^2 - 2x + 1,\ \ Q(x) = x^2 + 4x + 3,\ \ R(x) = 2x^3 + 4x^2 - x,\ \ T(x) = 4x^3 + 2x^2 - 6.$$

For polynomial T, calculate $T(1)$.

Exercise 3.2.2 let's consider functions f and g defined by: $f(x) = \dfrac{x-1}{x+1}$ and $g(x) = \dfrac{x-3}{x}$.

1. Determine the domain of definition of f and of g.
2. Determine functions $f+g$, $f-g$, $\dfrac{f}{g}$ as well as their domains of definition.

Exercise 3.2.3 For each of the functions defined above, specify the domain of definition and simplify the writing

$$f_1(x) = \frac{1}{1-x} - \frac{2}{1-x^2},\ \ f_2(x) = \frac{x-x^2}{x},\ \ f_3(x) = \frac{x^2-9}{x^2-2x-3}.$$
$$f_4(x) = \frac{x^2+3x-4}{2x^2+5x-7}.$$

Determine the limits in the following cases by detailing the reasoning

$$f_1(x) = \frac{1}{1-x} - \frac{2}{1-x^2},\ \ f_2(x) = \frac{x-x^2}{x},\ \ f_3(x) = \frac{x^2-9}{x^2-2x-3}.$$
$$f_4(x) = \frac{x^2+3x-4}{2x^2+5x-7}.$$

Exercise 3.2.4 We consider the functions

$$f(x) = \frac{1}{2}x^2 - x - \frac{3}{2},\ \ g(x) = -\frac{1}{2}x^2 + x + \frac{3}{2}.$$

Find by justifying the graph corresponding to each of the two functions f and g.

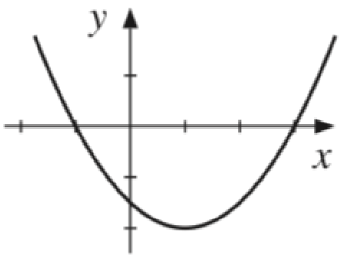


figure 1.

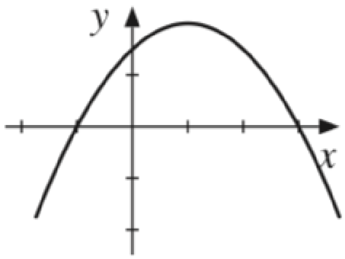


figure 2.

Exercise 3.2.5 ***Optimization problem of the second degree in economics.***
A company produces and sells Q units of a good at the price $P = 102 - 2Q$ and the production and the sale of Q units costs $C = 2Q + \frac{1}{2}Q^2$. Profit is given by the expression

$$F(Q) = PQ - C.$$

1. Develop the expression of the profit $F(Q)$.
2. Determine the value of Q which makes the profit $F(Q)$ maximum as well as the value of this maximum profit.

Chapter 4

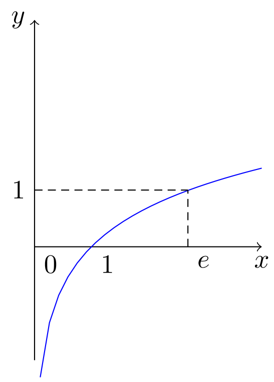
The Logarithm Function

4.1 Definition and Properties

Definition 4.1.1 *The natural logarithm function notated* ln *is defined on* $\mathbb{R}^{+*}$ *and*

$$\forall x > 0, \;\; \ln(x) = \int_1^x \frac{dt}{t}$$

The function ln *is the primitive of* $\frac{1}{t}$ *which is 0 at 1 therefore the function* ln *is continuous, differentiable on* $\mathbb{R}^{+*}$*.*



Proposition 4.1.2 *We have the following properties:*

1.
$$\forall x > 0, \ \ (\ln)'(x) = \frac{1}{x}, \ \ (\ln)''(x) = \frac{-1}{x^2}$$

2.
$$\ln(1) = 0 \ \ et \ \ \ln(e) = 1$$

3.
$$\lim_{x \to 0^+} \ln(x) = -\infty, \quad and \ \lim_{x \to +\infty} \ln(x) = +\infty$$

4.
$$\forall x > 0, \ \ \forall y > 0, \ \ \ln(xy) = \ln(x) + \ln(y)$$
$$\ln(\frac{1}{x}) = -\ln(x), \ \ \ln(\frac{y}{x}) = \ln(y) - \ln(x)$$
$$\forall x > 0, \ \ \forall \alpha \in \mathbb{R}, \ \ \ln(x^\alpha) = \alpha \ \ln(x).$$

4.2 Exercises

Exercise 4.2.1 1. Write as a function of $\ln 2$ numbers

$$A = \ln(\frac{e^2}{8}), \quad B = \ln(\frac{\sqrt{2}}{2}) + \ln(\frac{1}{4}).$$

2. Compare numbers

$$A = \ln 4 + \ln 3, \quad B = \ln 7.$$

Exercise 4.2.2 Calculate as a function of $\ln(3)$

$$A = 3\ln(e^2/9) + \ln(27) - \ln(1/3).$$

Exercise 4.2.3 Prove the following equalities:

$$a) \ \ \ln x - 2 = \ln\left(\frac{x}{e^2}\right), \quad b) \ \ \frac{1}{2}\ln x - \frac{3}{2}\ln x - \ln(x+1) = \ln\frac{x^2}{x+1},$$
$$c) \ \ 3 + 2\ln x = \ln(e^3 x^2), \quad d) \ \ \ln x - \ln y + 2\ln z = \ln\frac{xz^2}{y}.$$

Exercise 4.2.4 1. Demonstrate that function f given by $f(x) = \ln(x + \sqrt{x^2+1})$ is defined on $\mathbb{R}$.

2. Calculate $f(0)$.

3. Demonstrate that $f(1) + f(-1) = 0$ as for all real number x, $f(x) + f(-x) = 0$.

Exercise 4.2.5 We set, for all positive real numbers x ,

$$f(x) = 3\ln(x^4) - 4\ln(x^3).$$

1. Calculate $f(1)$ and $f(e)$.

2. Demonstrate that

$$\forall x > 0, \quad f(x) = 0.$$

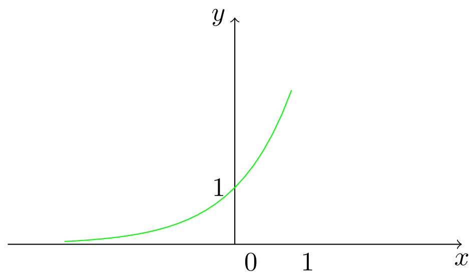
Chapter 5

The Exponential Function

5.1 Definition and Properties

Definition 5.1.1 *The exponential function is the inverse of the natural logarithm*

$$\forall x \in \mathbb{R}, \ \ \forall y > 0, \ \ y = exp(x) = e^x \Leftrightarrow x = \ln(y).$$



Proposition 5.1.2 *We have the following properties:*

1. $$\forall x \in \mathbb{R}, \ \ (exp)'(x) = e^x, \ \ (exp)''(x) = e^x.$$

2. $$e^0 = 1, \ \ e^1 = e$$

3. $$\lim_{x \to -\infty} e^x = 0, \ \ \lim_{x \to +\infty} e^x = +\infty$$

4. $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \ \ e^{x+y} = e^x e^y \ \ e^{-x} = \frac{1}{e^x}, \ \ (e^x)^y = e^{xy}.$$

5.2 Exercises

Exercise 5.2.1 Compare real numbers A and B

$$A = (e^3)^2 \text{ and } B = e^3 \times e^2.$$

then real numbers C and D

$$C = \frac{e^{-2}}{e^4} \text{ and } D = ((e^2)^6 \times e)^{-1}.$$

Exercise 5.2.2 For each of the functions defined below, specify the domain of definition

$$f_1(x) = \frac{1}{e^x - e^{-x}}, \quad f_2(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Exercise 5.2.3 Let's consider the function f defined on $\mathbb{R}$ by

$$f(x) = (e^x + e^{-x})^2 - e^x(e^x + e^{-3x}).$$

Prove that

$$\forall x \in \mathbb{R}, \quad f(x) = 2.$$

Exercise 5.2.4 Solve on $\mathbb{R}$ the following equations

1. $e^{3x+4} = 1/e$,
2. $e^{2x} - 1 = 0$,
3. $e^{2x} + 2e^x - 3 = 0$.

Exercise 5.2.5 Solve on $\mathbb{R}$ the following inequations

1. $e^{2x+1} < e$,
2. $3e^x + 1 > 0$,
3. $2e^{4x-2} - 4 < 0$.

Exercise 5.2.6 we consider the following function defined on $\mathbb{R}$

$$f(x) = \frac{6e^x}{e^x + 1}.$$

1. Calculate $f(0)$, $f(1)$ and $f(-1)$ as a function of e.
2. Demonstrate that for all real numbers x, $f(x) + f(-x)$ is constant.

Chapter 6

The Power Function

6.1 Definition and Properties

Definition 6.1.1 *Let $r \in \mathbb{R}$, the power function r is defined on $\mathbb{R}^{+*}$ by*

$$\forall x > 0, \quad x^r = e^{r \ln(x)}$$

The shape of the curves depends mainly on the value of r; as shown in figures 1 to 3:

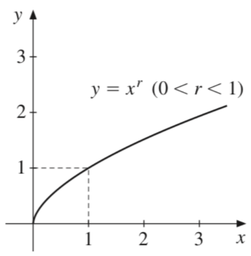


Figure 1

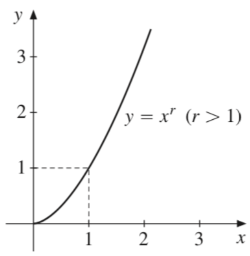


Figure 2

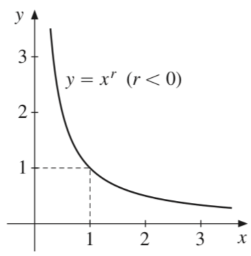


Figure 3

- If $0 < r < 1$, the graph is similar to figure 1 like the graph of $\sqrt{x}$.
- If $1 < r$, the graph is similar to figure 2, in particular if $r = 2$, it is half of a parabola $y = x^2$.
- Si $r < 0$, the graph is similar to figure 3, in particular if $r = -1$, it is a branch of the hyperbole $y = x^{-1} = \frac{1}{x}$.

Figure 4 shows, superimposed, the graphs of $y = x^r$ for various positive values of the exponent r.

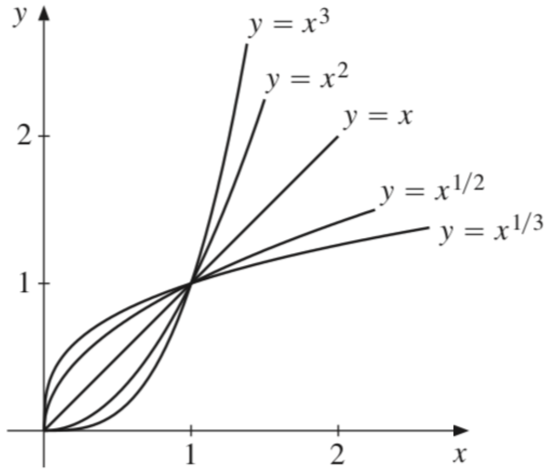


Figure 4

Proposition 6.1.2 *We have the following properties*

1. *The power function is defined, continuous and differentiable on* $\mathbb{R}^{+*}$

$$\forall x > 0, \ (x^\alpha)' = \alpha x^{\alpha-1}, \ (x^\alpha)'' = \alpha(\alpha-1)x^{\alpha-2},$$

2.

$$\lim_{x\to+\infty} x^\alpha = \begin{cases} +\infty & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ 0 & \text{if } \alpha < 0 \end{cases}$$

$$\lim_{x\to0^+} x^\alpha = \begin{cases} 0 & \text{if } \alpha > 0 \\ 1 & \text{if } \alpha = 0 \\ +\infty & \text{if } \alpha < 0 \end{cases}$$

Proposition 6.1.3 Relative rates of growth *Let* $\alpha \in \mathbb{R}^{+*}$ *and* $\beta \in \mathbb{R}^{+*}$, *we have*

1.

$$\lim_{x\to+\infty} \frac{e^{\beta x}}{x^\alpha} = +\infty,$$

2.

$$\lim_{x\to+\infty} \frac{(\ln(x))^\alpha}{x^\beta} = 0$$

We deduce that

Proposition 6.1.4

$$\lim_{x\to-\infty} |x|^{\alpha} e^{\beta x} = 0, \quad \lim_{x\to 0^+} |\ln(x)|^{\alpha} x^{\beta} = 0$$

Example 6.1.5

$$\lim_{x\to+\infty} \frac{e^{2x}}{x^3} = +\infty, \;\; \textit{and} \;\; \lim_{x\to+\infty} \frac{\ln x}{x^{3/2}} = 0.$$

Remark Consider $a > 0$. It should be noted that the two functions

$$f(x) = a^x \text{ and } g(x) = x^a$$

are fundamentally different. The second belongs to the family of power functions. While the function $f(x)$ is an exponential function that can be written

$$f(x) = a^x = \exp\left(x \ln(a)\right).$$

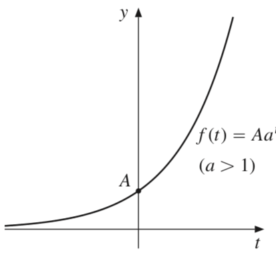


Figure 1 Graphique de $f(t) = Aa^t$ $(a > 1)$

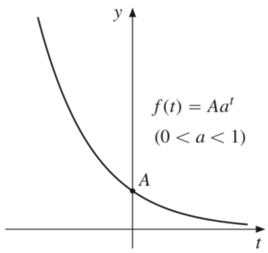


Figure 2 Graphique de $f(t) = Aa^t$ $(0 < a < 1)$

6.2 Exercises

Exercise 6.2.1 Let's consider the two following functions, defined on $\mathbb{R}$ by

$$f(x) = \sqrt{4x^2+9} + 2x, \quad g(x) = \sqrt{4x^2+9} - 2x.$$

1. Determine the limit of f when $x \to +\infty$.
2. Demonstrate that the function g can be written as

$$\forall x \in \mathbb{R}, \quad g(x) = \frac{9}{\sqrt{4x^2+9} + 2x}.$$

3. Deduce the limit of g when $x \to +\infty$.

Exercise 6.2.2 Let's consider the function

$$\forall x > 0, \quad f(x) = x^x = e^{x \ln x}.$$

1. Calculate $f(1), f(2), f(3)$.
2. Determine the limit of f at 0, then at $+\infty$.

Exercise 6.2.3 Let's consider the function

$$\forall x > 0, \quad f(x) = x^{1/x}.$$

1. Calculate $f(1), f(2)$.
2. Write the function f in exponential form.
3. Determine the limit of f at 0, then at $+\infty$.

Exercise 6.2.4 Let's consider the function

$$\forall x > 1, \quad f(x) = 4x^3 + \sqrt{\ln(x)} - e^{4x}.$$

1. Calculate $f(1), f(2)$.
2. Write f factoring out e^{4x}.
3. Determine the limit of f at $+\infty$.

Exercise 6.2.5 Let's consider the function defined on a domain D

$$\forall x \in D, \quad f(x) = \frac{\ln(1+e^x)}{x}.$$

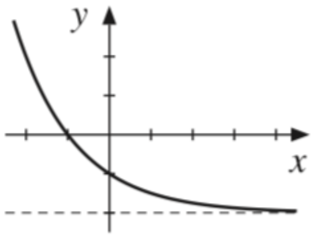
1. Prove that for all real numbers x, $e^x + 1$ is positive.
2. Deduce the domain of definition D of f.
3. Determine the limit of f at 0.
4. Determine the limit of f at $-\infty$.
5. By factoring out e^x in the logarithm, demonstrate that

$$\forall x \in D, \quad f(x) = 1 + \frac{\ln(1+e^{-x})}{x}.$$

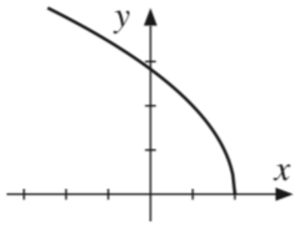
6. Deduce the limit of f at $+\infty$.

Exercise 6.2.6 Match each of the graphs A, B and C with one of the functions f, g or h:

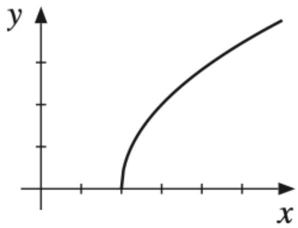
$$f(x) = \sqrt{x-2}, \quad g(x) = \sqrt{2-x}, \quad h(x) = \left(\tfrac{1}{2}\right)^x - 2,$$



A



B



C

Chapter 7

Derivatives

“ An important topic in many scientific disciplines, including economics, is the study of how quickly quantities change over time. In order to compute the future position of a planet, to predict the growth in population of a biological species, or to estimate the future demand for a commodity, we need information about **rates of change**.
The concept used to describe the **rate of change of a function** is the derivative, which is the central concept in mathematical analysis. This chapter defines the derivative of a function and presents some of the important rules for calculating it.
Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) discovered most of these general rules independently of each other. This initiated differential calculus, which has been the foundation for the development of modern science. It has also been of central importance to the theoretical development of modern economics.

7.1 Calculation of derivative of function of one variable

7.1.1 Definition

Definition 7.1.1 *Let a be a real number and let f be a function defined on a set D which includes a. We name rate of change of f at a, written $\theta_a(x)$, the quotient function defined by*

$$\forall x \in D \setminus \{a\}, \ \ \theta_a(x) = \frac{f(x) - f(a)}{x - a}.$$

The function f is differentiable at a if the function θ_a has a limit when x approaches a. We write this limit as $f'(a)$.
we say that f is differentiable on D if f is differentiable at each point of D and we denote by f' the derivative function of f.

Example 7.1.2 *Let's consider function $f(x) = x^2$ and let a be a real number, then for all real number $x \neq a$, we have*

$$\theta_a(x) = \frac{x^2 - a^2}{x - a} = \frac{(x-a)(x+a)}{x - a} = x + a.$$

The limit of $\theta_a(x)$ is $2a$ when x approaches a therefore f is differentiable at a and we get back to the formula known to us

$$\forall a \in \mathbb{R}, \ \ f'(a) = 2a.$$

Example 7.1.3 *Limit computation using the rate of change:*
Consider $l = \lim_{x \to 1} \frac{x^4 - 1}{x - 1}$. *To compute* l, *we set* $f(x) = x^4$ *and we write*

$$l = \lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \theta_1(x),$$

where $\theta_1(x)$ *is the rate of change of* f *at 1. Since* f *is differentiable at* 1, *we deduce that*

$$l = \lim_{x \to 1} \theta_1(x) = f'(1) = 4.$$

7.1.2 Equation of Tangent

Definition 7.1.4 *Let* a *be a real number and let* f *be a function defined on a set* D *which includes* a. *We assume that function* f *is differentiable at* a, *the tangent to the graph of* f *at point* $A = (a, f(a))$ *is the line passing through point* A , *with direction vector* $(1, f'(a))$.
An equation of tangent is

$$y = f'(a)(x - a) + f(a).$$

$f'(a)$ *is the gradient of the tangent.*
We say that f *is differentiable on* D *if* f *is differentiable at each point of* D *and we denote by* f' *the derivative function of* f.

7.1.3 Formulas

Function f	Derivative f'	Interval
x^n, $n \in \mathbb{N}$	nx^{n-1}	$\mathbb{R}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$	$]0, +\infty[$
$\dfrac{1}{x}$	$\dfrac{-1}{x^2}$	$\mathbb{R}^*$
$\ln x$	$\dfrac{1}{x}$	$]0, +\infty[$
e^x	e^x	$\mathbb{R}$
x^α, $\alpha \in \mathbb{R}$	$\alpha x^{\alpha-1}$	$]0, +\infty[$.

Function f	Derivative f'
$u + v$	$u' + v'$
uv	$u'v + uv'$
$\dfrac{1}{u}$	$\dfrac{-u'}{u^2}$
$\dfrac{u}{v}$	$\dfrac{u'v - uv'1}{v^2}$
$\ln(\lvert u \rvert)$.	$\dfrac{u'}{u}$
e^u	$e^u \times u'$
$(u)^\alpha$	$\alpha\,(u)^{\alpha-1} \times u'$
$\sqrt{u} = (u)^{1/2}$	$\frac{u'}{2\sqrt{u}}$

7.2 Calculation of derivative of function of two variables

7.2.1 Definition of a function of two variables

Definition 7.2.1 *A function f of two real variables x and y which the definition domain is D is a rule which a rule which associates a unique real number $f(x;y)$ to each couple (x,y) of D.*

Exemple 1 *Consider $f(x,y) = x^2 + y^2$, defined on $D = \{(x,y), x \in \mathbb{R},\ y \in \mathbb{R}\} = I\!R^2$.*

Exemple 2 *Consider $f(x,y) = \sqrt{x-1} + \sqrt{y}$. To be able to calculate $\sqrt{x-1}$ and $\sqrt{y}$, we must have $x \geq 1$ and $y \geq 0$. The set of points with coordinates (x,y) which satisfy these inegqualities is given by Figure 1.*

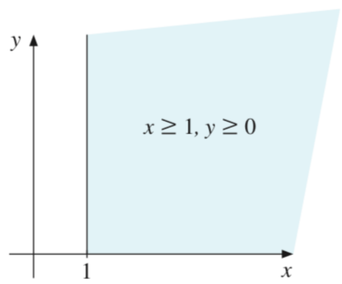


Figure 1

7.2.2 The first partial derivatives of the functions of two variables

1. $\frac{\partial f}{\partial x}(x,y)$ denotes the derivative of $f(x,y)$ with respect to x when y is held constant. We say that $\frac{\partial f}{\partial x}(x,y)$ is the partial derivative of f with respect to x.
2. $\frac{\partial f}{\partial y}(x,y)$ denotes the derivative of $f(x,y)$ with respect to y when x is held constant.

   We say that $\frac{\partial f}{\partial y}(x,y)$ is the partial derivative of f with respect to y.

Exemple 3 *Calculate the partial derivatives of the following functions*

$$(a)\ \ f(x,y) = x^3y + x^2y^2 + x + y^2, \quad (b)\ \ g(x,y) = \frac{xy}{x^2+y^2}.$$

Solution:
(a) We get

$$\frac{\partial f}{\partial x}(x,y) = 3x^2y + 2xy^2 + 1 \text{ y is held constant}.$$

$$\frac{\partial f}{\partial y}(x,y) = x^3 + 2x^2y + 2y \text{ x is held constant}.$$

(b) For this function, we use the rule to derivate a quotient

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(x^2+y^2) - xy(2x)}{(x2+y^2)^2} = \frac{y^3 - x^2 y}{(x2+y^2)^2},$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{x^3 - y^2 x}{(x2+y^2)^2}.$$

7.2.3 The second partial derivatives of the functions of two variables

The first partial derivatives are functions of two variables that we may consider partially derivating with respect to x and to y, on condition that these new partial derivatives exist. They are called partial second or second order derivative of $f(x,y)$ and are four in number. They are noted

$$\boxed{\begin{array}{cc} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y}, \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{y \partial x}. \end{array}}$$

Exemple 4 *Consider the example 3 (a), we have*

$$\frac{\partial^2 f}{\partial x^2} = 6x^2 y + 2y^2,\ \frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 4xy,\ \frac{\partial^2 f}{\partial y^2} = 2x^2 + 2,\ \frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 4xy.$$

7.3 Exercises

Exercise 7.3.1 Considering the rate of change at 0, demonstrate that function f defined on $[0,+\infty[$ by $f(x)=\sqrt{x}$ is not differentiable at 0.

Exercise 7.3.2 using the notion of rate of change, determine the following limits:

$$a)\lim_{x\to 1}\frac{x^4-1}{x-1},\quad b)\lim_{x\to 1}\frac{\ln(x)}{x-1},\quad c)\lim_{x\to 0}\frac{e^x-1}{x},\quad d)\lim_{x\to 0}\frac{\ln(1+x)}{x}.$$

Exercise 7.3.3 After having specified the set of differentiability of the following functions, determine their derivative :

$$\begin{array}{ccc} f_1(x)=2x^2-4x+1, & f_2(x)=-x^3+2x^2+\frac{1}{x}, & f_3(x)=\sqrt{x+3}, \\ f_4(x)=3-\dfrac{7}{x+1}, & f_5(x)=(2x+3)^{10}, & f_6(x)=\dfrac{1}{(2x+3)^{10}}, \\ f_7(x)=(-4x^2+3)^3, & f_8(x)=x\sqrt{4-x^2}, & f_9(x)=\dfrac{x-1}{x+1}, \\ f_{10}(x)=\dfrac{-3x^2+7x-1}{x-4}, & f_{11}(x)=8\sqrt{\frac{1}{x}}, & f_{12}(x)=(x^2+x+1)^7. \end{array}$$

$$\begin{array}{ccc} g_1(x)=3^x x^3, & g_2(x)=x\ln(x)-x, & g_3(x)=\ln(2x+3), \\ g_4(x)=\dfrac{\sqrt{x}}{1+e^{2x}}, & g_5(x)=e^{2x+3}, & g_6(x)=\ln(\dfrac{1}{1+x^2}), \\ g_7(x)=e^{-x^2/2}, & g_8(x)=(2x^2+1)^{3/2}, & g_9(x)=x^x=e^{x\ln(x)}, \\ g_{10}(x)=\ln(e^x+1), & g_{11}(x)=\sqrt{\dfrac{\ln x}{x}}, & g_{12}(x)=\dfrac{e^{-x}}{2-x}. \end{array}$$

Exercise 7.3.4 *Calculate the partial derivatives of the following functions:*

$$\begin{array}{lll} F_1(x,y)=3x+y, & F_2(x,y)=x^3y^2, & F_3(x,y)=\sqrt{x^2+y^2}, \\ F_4(x,y)=\dfrac{x}{y}, & F_5(x,y)=\frac{x-y}{x+y}, & F_6(x,y)=y\ln(x), \\ G_1(k,x)=x^2+e^{2k}, & G_2(a,b)=a^b & G_3(y,t)=ty^2+e^{ty}. \end{array}$$

Exercise 7.3.5 Let $k\in\mathbb{R}^{+*}$, we define on $\mathbb{R}$ the function g_k by

$$\forall x\in\mathbb{R},\ \ g_k(x)=e^{-kx^2}.$$

1. Demonstrate that g_k is differentiable and calculate its derivative. Deduce the variation table of g_k.
2. Calculate g_k'' and solve the equation $g_k''(x)=0$.
3. Demonstrate that for each positive real numbers h and k

$$h\le k\iff g_h\ge g_k.$$

4. Trace the curve of g_1.

Chapter 8

Calculation of Integrals

The main topic of the preceding chapter was differentiation, which can be directly applied to many interesting economic problems. Economists, however, especially when doing statistics, often face the mathematical problem of finding a function from information about its derivative. This process of reconstructing a function from its derivative can be regarded as the "inverse" of differentiation. Mathematicians call this integration. There are simple formulas that have been known since ancient times for calculating the area of any triangle, and so of any polygon that, by definition, is entirely bounded by straight lines. Over 4000 years ago, however, the Babylonians were concerned with accurately measuring the area of plane surfaces, like circles, that are not bounded by straight lines. Solving this kind of area problem is intimately related to integration, as will be explained in Section 8.1.3.

8.1 Definition and Properties

8.1.1 Primitives

Definition 8.1.1 *Let f be a continuous function on an interval I (non-empty and not reduced to one point) of $\mathbb{R}$, a primitive of f on I is a function F for which*

$$\forall x \in I, \quad F'x) = f(x).$$

In the table below, you can find the primitives to be learned:

Function	Interval	Primitive
$\alpha \in \mathbb{R}^*$, $e^{\alpha x}$	$\mathbb{R}$	$\frac{1}{\alpha}e^{\alpha x} + c$
$a \in \mathbb{R}^{+*} \setminus \{1\}$, $a^x = e^{xLn(a)}$	$\mathbb{R}$	$\frac{1}{\ln(a)}a^x + c$
$\alpha \in \mathbb{R}, \alpha \neq -1$, x^α	depends on α	$\frac{x^{\alpha+1}}{\alpha+1} + c$
$1/x$	$]-\infty, 0[$ or $]0, +\infty[$	$\ln(\|x\|) + c$

8.1.2 Definite integral

Definition 8.1.2 *Let f be a continuous function on an interval I (non-empty and not reduced to one point) of $\mathbb{R}$ and any primitive F of f on I.*
Therefore, $\forall(a,b) \in I^2$, the real number $F(b)-F(a)$ is independent from the primitive F chosen and is called **definite integral of f between bounds a and b.**
We denote it by: $\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$ which is enunciated as sum from a to b to $f(x)dx$.

Remark:
The numbers a and b are called, respectively, the lower and upper limit of integration. The variable x is a dummy variable in the sense that it could be replaced by any other variable that does not occur elsewhere in the expression. For instance,

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du = \int_a^b f(\xi)d\xi$$

are all equal to $F(b) - F(a)$.
But do not write anything like $\int_a^y f(y)dy$, with the same variable as both the upper limit and the dummy variable of integration, because that is meaningless.

8.1.3 Geometrical Interpretation of the Notion of Definite Integral

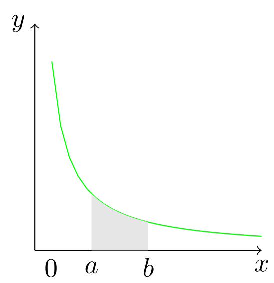
Let's assume the plane referenced to an orthogonal frame (generally orthonormal). We choose as surface area unit: the area of the rectangle on the unit vectors of the frame and we construct the curve representing f on $[a, b]$. (we assume here $a < b$).

In particular, if f **is a constant function on** $[a, b]$ such that: $\forall x \in [a, b],\ f(x) = c$ we have $\int_a^b f(x)dx = c(b-a)$. this result is interpreted as the area of a rectangle and we shall admit that this result can be generalised as follows: If f is a positive function, continuous on $[a, b]$ with $a < b$, then $\int_a^b f(x)dx$ is the area $\mathcal{A}$ of domain $\mathcal{D}$ of the plane defined by $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 / a \leq x \leq b \text{ et } 0 \leq y \leq f(x)\}$.

Example 8.1.3 *Let's consider the function $f(x) = 1/x$ defined on $\mathbb{R}^*$, we have for $0 < a < b$*

$$\int_a^b f(x)dx = [\ln(x)]_a^b = \ln(b) - \ln(a) = \ln(b/a).$$

This integral corresponds to the grey area in the figure below.



Remark 8.1.4 *In this writing, x is a dummy variable: $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(u)du$.*

Remark 8.1.5 *By definition: $\int_a^a f(x)dx = 0$ and $\int_a^b f(x)dx = -\int_b^a f(x)dx$.*

8.2 Properties of the Definite Integral

8.2.1 Chasles Relation

Theorem 8.2.1 Chasles Relation. *Let f be a continuous functions on an interval I. Then,*

$$\forall (a,b,c) \in I^3, \quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

The order of real numbers a, b, c, is not important, however we can't get outside of the interval I.

Example 8.2.2 *Calculate* $I_1 = \displaystyle\int_{-1}^{2} |x|\, dx.$

8.2.2 Linearity

Theorem 8.2.3 *If f and g are two continuous functions on an interval I , then:*

1. $\displaystyle\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx$ *where α is a constant.*
2. $\displaystyle\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$
3. *Repeated use of these properties yields the general rule:*

$$\forall (\alpha, \beta) \in \mathbb{R}^2, \forall (a,b) \in I^2, \int_a^b (\alpha f(x) + \beta g(x))dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Example 8.2.4 *Calculate* $I_2 = \displaystyle\int_1^2 \frac{2x^5 - 3x^2 + \sqrt{x}}{x^{3/2}} dx.$

Remark 8.2.5 *Do not confuse Chasles relation and linearity. Chasles relation involves only one function and we "divide" the interval of integration. Linearity involves only one interval and we "fraction" the function.*

8.2.3 Integration by Parts

This method is based on a very simple idea, to use the derivative of a product, and gives a very powerful result which allows to calculate various integrals.

Theorem 8.2.6 *Let u and v* **be two functions of class** C^1 *on an interval I, then:*

$$\forall (a,b) \in I^2, \int_a^b u(x)v'(x)dx = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)dx.$$

This theorem is interesting when the integral on the right-hand side is easier to calculate compared than the integral on the left-hand side.

Example Calculate $I_4 = \displaystyle\int_0^1 xe^x dx$.

Solution: we must write the integrand in the form $u(x)v'(x)$. Let $u(x) = x$ and $v'(x) = e^x$, implying that $v(x) = e^x$ and $u(x)v'(x) = xe^x$.

$$\begin{array}{ccccccccc} \displaystyle\int_0^1 & x & e^x dx & = & x & e^x - \displaystyle\int_0^1 & 1 & \times & e^x dx \\ & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \\ & u(x) & v'(x) & & u(x) & v(x) & u'(x) & & v(x) \end{array}$$

We can also use an integration by parts, in the special case where there is only one function, by setting $v'(x) = 1$.

Example 8.2.7 *Calculate* $I_5 = \displaystyle\int_2^3 ln(x)dx$.

8.3 Exercises

Exercise 8.3.1 Calculate the following integrals:

$$I_1 = \int_0^3 x^2 - 3x + 2\,dx,\ \ I_2 = \int_1^2 (x - \frac{1}{\sqrt{x}})x^2 dx,$$

$$I_3 = \int_{-1}^2 |x|\,dx,\ \ I_4 = \int_1^2 \frac{2x^5 - 3x^2 + \sqrt{x}}{x^{3/2}} dx,$$

$$I_5 = \int_0^1 2^{3x} e^x dx \text{ et } I_6 = \int_{-1/2}^{1/2} \frac{2}{x^2-1} dx,$$

for I_6, we can observe that $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{1+x}$.

Exercise 8.3.2 Calculate the following integral using an integration by parts.

$$I_1 = \int_1^2 x^\alpha \ln(x)dx,\ \ (\alpha \neq -1),\ \ I_2 = \int_0^1 x^2 2^x dx.$$

Exercise 8.3.3 Knowing that function f is C^2 on $I\!R$, that $f(1) = 2, f(4) = 7, f'(1) = 5$ and $f'(4) = 3$, calculate the integral $I = \int_1^4 x f''(x)dx$.

Exercise 8.3.4 Let's consider the fonctions f and h defined on $I\!R$ by :

$$\forall x \in I\!R,\ \ f(x) = \frac{1}{1+e^x}\ \ ,\ \ h(x) = \frac{e^x}{(e^x+1)^2}.$$

1. Prove that
$$\forall x \in I\!R,\ \ f'(x) = h(x).$$

2. Let A be a real, by using the answer of question 1

   (a) Compute the integral according to A : $\displaystyle\int_0^A h(x)dx$.

   (b) Then prove that the limit $I = \displaystyle\int_0^{+\infty} h(x)dx = \lim_{A\to+\infty} \int_0^A h(x)dx$ exists and compute his value.

Chapter 9

Reviewing: study of function

Exercise 9.0.1 *Final exam January 2020*
We consider the function f defined by :

$$\forall x \in \mathbb{R}, \ \ f(x) = x - 1 + (x^2 + 2)e^{-x}.$$

1. *Calculate $f(0)$.*

2. *Determine the limits of $f(x)$ when x tends to $+\infty$ and when x tends to $-\infty$.*

3. *Prove that*
$$\forall x \in \mathbb{R}, \ \ f'(x) = 1 - (x^2 - 2x + 2)e^{-x}.$$

4. *Calculate using the rate of change* $\lim\limits_{x \to 0} \dfrac{x - 2 + (x^2 + 2)e^{-x}}{x}$.

5. *We consider the function h defined*
$$\forall x \in \mathbb{R}, \ \ h(x) = 1 - (x^2 - 2x + 2)e^{-x}.$$
   (a) *Calculate $h(0)$.*
   (b) *Calculate the derivative function of h.*
   (c) *Deduce the variation table of h.*
   In the following, we admit that** $h(x) = 0$ **has an unique solution noted** α**.
   and** $\alpha \in]0, 1[$**.

   (d) *Expressing f' as a function of h, deduce the variation table of f.*
   (e) *Prove that $f(\alpha) = \alpha(1 + 2e^{-\alpha})$*

6. (a) *Prove that the second derivative function of f is*
$$\forall x \in \mathbb{R}, \ \ f''(x) = (x - 2)^2 e^{-x}.$$
   (b) *What can we deduce about the convexity of the function f.*

7. *Give an equation of the tangent at the point of abscissa $x = \alpha$ to the graph of f.*

8. *Prove that α is a global minimum of f.*

9. *Draw the graph of f by representing the tangent at the point of abscissa $x = \alpha$ (indication: $f(\alpha) \sim 0.85$).*

Exercise 9.0.2 *Final Exam january 2019*
We consider the function f defined by :

$$\forall x \in \mathbb{R}^{+*}, \quad f(x) = 2\ln(x) - \frac{\ln(x)}{x} + \frac{1}{x}.$$

1. *Calculate $f(1)$.*
2. *Determine the limits of $f(x)$ when x tends to $+\infty$ and when x tends to 0.*
   For the limit at 0, we can factor $\ln(x)/x$.
3. *Prove that*
   $$\forall x \in \mathbb{R}^{+*}, \quad f'(x) = \frac{2x - 2 + \ln(x)}{x^2}.$$
4. *We consider the function h defined by*
   $$\forall x \in \mathbb{R}^{+*}, \quad h(x) = 2x - 2 + \ln(x).$$
   (a) *Calculate $h(1)$.*
   (b) *Calculate the derivative function of h.*
   (c) *Expressing f' as a function of h, give the variation table of f.*
5. (a) *Prove that the second derivative function of f is*
   $$\forall x \in \mathbb{R}^{+*}, \quad f''(x) = \frac{1}{x^3}(5 - 2x - 2\ln(x)).$$
   (b) *We admit that the equation $f''(x) = 0$ has a unique solution noted a in $\mathbb{R}^{+*}$ and that $1 < a < 2$. Deduce that f is convex on $]0, a]$ and concave on $[a, +\infty[$. We can approximate e by 2.7.*
6. *Give an equation of the tangent line at $x = 1$ of f.*
7. *Draw the graph of f and represent the tangent at the point $x = 1$.*
8. (a) *We set*
   $$\forall x \in \mathbb{R}^{+*}, \quad F(x) = (2x+1)\ln(x) - 2x - \frac{1}{2}\ln^2(x).$$
   Calculate the derivative function F' of F.

Exercise 9.0.3 *Final Exam January 2018*
We consider the function f defined on $I\!R$ by :

$$\forall x \in I\!R, \quad f(x) = e^{2x} - 6e^x + 9.$$

We give the approximate values of the logarithm in 2 and 3

$$\ln(2) \simeq 0,69 \text{ and } \ln(3) \simeq 1,1.$$

1. *Justify the the definition domain of f is $I\!R$ and calculate $f(\ln 3)$.*
2. *Determine the limits of $f(x)$ when x tends to $+\infty$ and when x tends to $-\infty$.*

3. *Prove that* $\forall x \in \mathbb{R},\ \ f'(x) = 2e^x(e^x - 3)$. *Deduce the variation table of* f.
4. *Determine the real numbers* x *such that* $f(x) = 9$. *We will give an approximate value of these reals using the two approximate values given at the start of the problem.*
5. *Determine the interval(s) over which* f *is convex or concave.*
6. *Give an equation of the tangent at the point of abscissa* $x = 0$ *to the graph of* f.
7. *Draw the graph of* f *by representing the tangent at the abscissa point* $x = 0$.